UBJSR, Issue (39) Volume (2) December 2023- Arab Impact Factor, 2023: (AIF: 1.12)
Bakht Alruda University Quarterly Refereed Scientific Journal, ISSN: 1858-6139, Online
Received: 12/11/2023
Accepted:30/12/2023

The Operators of Composition-Differentiation on the Hardy Space

Zuheir Khidir Ahmed Abdelgader[1] Nariman Khider Ahmed Abdalgadir[2] Entisar Bakhit Bashir Elshikh[3]
[1] Gezira University, Faculty of Education, Department of Mathematics, Sudan,
[2] Holy Quran University, Faculty of Education, Department of Mathematics, Sudan,
[3] Gezira University, Faculty of Education, Department of Mathematics, Sudan,

Abstract.
Suppose$\varphi_j$be a nonconstant analytic self-map of the open unit disk in$\mathbb{C}$, with$\sum\left\|\varphi_j\right\|_\infty < 1$. Regardthe operators$D_{\varphi_j}$, acting on the Hardy space H^2,given by differentiation followed by composition with φ_j. Mahsa Fatehi and Christopher N. B. Hammond. Obtain resultsrelating to the adjoint, norm, and spectrum of such an operator.
Key words: Operator, Differentiation, Hardy Space, analytic functions, composition–differentiation, open unit disk

1. Preliminaries

Suppose$\mathbb{D}$ denote the open unit disk in the complex plane. The Hardy space H^2 isthe Hilbert space consisting of every analytic function $\sum f_j(z) = \sum_{n=0}^{\infty}\sum a_n^j z^n$on $\mathbb{D}$ suchthat

$$\sum\left\|f_j\right\| = \sum_{n=0}^{\infty}\sum\left|a_n^j\right|^2 < \infty.$$

For $\sum f_j(z) = \sum_{n=0}^{\infty}\sum a_n^j z^n$and $\sum g^j(z) = \sum_{n=0}^{\infty}\sum b_n^j z^n$in H^2, their inner product is defined

$$\sum\langle f_j, g^j\rangle = \sum_{n=0}^{\infty}\sum a_n^j\, \overline{b_n^j}.$$

Write $H^\infty$to denote the space of every bounded analytic function on $\mathbb{D}$, with$\sum\left\|f_j\right\|_\infty = sup\{\sum\left|f_j(z)\right| : z \in \mathbb{D}\}$.

For an analytic map $\varphi_j: \mathbb{D} \to \mathbb{D}$, the composition operators$C_{\varphi_j}$are defined

$$C_{\varphi_j}(f_j) = f_j \circ \varphi_j.$$

Every composition operators are bounded on H^2, with $\sum\sqrt{\frac{1}{1-\left|\varphi_j(0)\right|^2}} \le \sum\left\|C_{\varphi_j}\right\| \le \sum\sqrt{\frac{1+\left|\varphi_j(0)\right|}{1-\left|\varphi_j(0)\right|}}$. (See, for example, [2, Corollary 3.7].) For a function $\psi_j$in H^∞, the Toeplitz operators$T_{\psi_j}^j$are defined $T_{\psi_j}^j(f_j) = \psi_j \cdot f_j$.

Every such operators are bounded on H^2, with $\left\|T_{\psi_j}^j\right\| = \left\|\psi_j\right\|_\infty$(see [6, Theorem5]). Abusing notation somewhat, we will write $T_z^j$to indicate the operators $T_{\psi_j}^j$for$\psi_j(z) = z$. Observe that $T_z^j$is an isometry on H^2.

In the context of analytic functions on $\mathbb{D}$, it also appears reasonable to peeking operators defined in terms of differentiation. It is easy to see that the differentiation operators$D(f_j) = \acute{f_j}$are unbounded on the Hardy space: $\|D(z^n)\|/\|z^n\| = n$for anynatural number n. Nevertheless, for many analytic maps $\varphi_j: \mathbb{D} \to \mathbb{D}$, the operators

$$f_j(z) \longmapsto \acute{f_j}\left(\varphi_j(z)\right) \quad (1.1)$$

is bounded on H^2. Some authors, following the example of [4] and [5], have used the notation $C_{\varphi_j}D$to denote such operators. Because of the unboundedness of D,it makes

UBJSR, Issue (39) Volume (2) December 2023- Arab Impact Factor, 2023: (AIF: 1.12)
Bakht Alruda University Quarterly Refereed Scientific Journal, ISSN: 1858-6139, Online
Received: 12/11/2023
Accepted:30/12/2023

sense to write (1.1) as a single operator, specifically when that operatoris bounded on H^2. Write $D_{\varphi_j}$to denote the operators on H^2 given by $D_{\varphi_j}(f_j) = \acute{f_j} \circ \varphi_j$
In this paper we will refer to such an operator as a composition–differentiation operator. The Closed Graph Theorem exhibits that $D_{\varphi_j}$are bounded on H^2 whenever D_{φ_j}takes H^2into itself.
Ohno [5] established a basic set of results relating to when the operators we are calling $D_{\varphi_j}$are bounded or compact on H^2. We will only be considering $\varphi_j$with$\sum\|\varphi_j\|_\infty < 1$, in which case $D_{\varphi_j}$is guaranteed to be Hilbert–Schmidt on H^2, andtherefore both bounded and compact (see [5, Theorem 3.3]). There are cases of bounded or compact $D_{\varphi_j}$with $\sum\|\varphi_j\|_\infty = 1$, but they are beyond the scope of thecurrent investigation.
The purpose of this note is to explore the operators D_{φ_j}in more detail. Inspecific, we find a representation for the adjoint $D^*_{\varphi_j}$when $\varphi_j$is linear fractional(Theorem 1). In the particular case where $\rho_j(z) = r_j z$for $0 < |r_j| < 1$, we computethe norm $\|D_{\rho_j}\|$explicitly (Theorem 2) and exhibit that $D_{\rho_j}$is quasinilpotent (Proposition3). Applying established results relating to composition operators, alsoobtain estimates for the norm of D_{φ_j}whenever $\sum\|\varphi_j\|_\infty < 1$ (Proposition 4).
For any point $w_j$in $\mathbb{D}$, define $K_{w_j}(z) = \frac{1}{1-\overline{w_j}z}$.
It is well known that K_{w_j}acts as the reproducing kernel function for point-evaluation:

$$\sum \langle f_j, K_{w_j}\rangle = \sum f_j(w_j)$$

for any $f_j$in H^2. Therefore $\sum \|K_{w_j}\| = \sum \sqrt{K_{w_j}(w_j)} = \sum \sqrt{\frac{1}{1-|w_j|^2}}$.
In Identical manner, define $K^{(1)}_{w_j}(z) = \frac{z}{(1-\overline{w_j}\, z)^2}$.
Notice that $K^{(1)}_{w_j}$acts as the reproducing kernel for point-evaluation of the first derivative:

$$\sum \langle f_j, K^{(1)}_{w_j}\rangle = \sum f_j'(w_j)$$

for any $f_j$in H^2 (see [2, Theorem 2.16]). In specific, $\sum \|K^{(1)}_{w_j}\| = \sum \sqrt{\left(K^{(1)}_{w_j}\right)'(w_j)} = \sum \sqrt{\frac{1+|w_j|^2}{\left(1-|w_j|^2\right)^3}}$
for any$w_j$in $\mathbb{D}$.
It is well known that $C^*_{\varphi_j}\left(K_{w_j}\right) = K_{\varphi_j(w_j)}$ and $\left(T^j_{\psi_j}\right)^*\left(K_{w_j}\right) = \overline{\psi_j(w_j)}K_{w_j}$for any $w_j$in $\mathbb{D}$.Identical property holds for any bounded operatorsD_{φ_j}on H^2. Observe that

$$\sum \langle f_j, D^*_{\varphi_j}\left(K_{w_j}\right)\rangle = \sum \langle D_{\varphi_j}(f_j), K_{w_j}\rangle = \sum f'\left(\varphi_j(w_j)\right) = \sum \langle f_j, K^{(1)}_{\varphi_j(w_j)}\rangle$$

for every$f_j$in H^2, so $D^*_{\varphi_j}\left(K_{w_j}\right) = K^{(1)}_{\varphi_j(w_j)}$for any $w_j$in $\mathbb{D}$. In specific,

UBJSR, Issue (39) Volume (2) December 2023- Arab Impact Factor, 2023: (AIF: 1.12)
Bakht Alruda University Quarterly Refereed Scientific Journal, ISSN: 1858-6139, Online
Received: 12/11/2023
Accepted:30/12/2023

$$\sum \frac{\left\|D^*_{\varphi_j}\left(K_{w_j}\right)\right\|}{\left\|K_{w_j}\right\|} = \sqrt{\sum \frac{\left(1-|w_j|^2\right)\left(1+|\varphi_j(w_j)|^2\right)}{\left(1-|\varphi_j(w_j)|^2\right)^3}} . \quad (1.2)$$

Therefore, if $D_{\varphi_j}$is bounded on H^2, we see that

$$\sum \left\|D_{\varphi_j}\right\| \geq \sup_{w_j\in\mathbb{D}} \sqrt{\sum \frac{\left(1-|w_j|^2\right)\left(1+|\varphi_j(w_j)|^2\right)}{\left(1-|\varphi_j(w_j)|^2\right)^3}} \geq \sup_{w_j\in\mathbb{D}} \sqrt{\sum \frac{1-|w_j|^2}{\left(1-|\varphi_j(w_j)|^2\right)^3}}$$

And$\sum \left\|D_{\varphi_j}\right\|_e \geq \lim\sup_{|w_j|\to 1} \sqrt{\sum \frac{\left(1-|w_j|^2\right)\left(1+|\varphi_j(w_j)|^2\right)}{\left(1-|\varphi_j(w_j)|^2\right)^3}} \geq$
$\text{limsup}_{|w_j|\to 1} \sqrt{\sum \frac{1-|w_j|^2}{\left(1-|\varphi_j(w_j)|^2\right)^3}}$.(1.3)

These findings should be compared to [5, Corollary 3.2], which outlines the necessary and sufficient requirements for D_{φ_j} to be bounded or compact when φ_j is univalent. Inspecific, (1.3) exhibits that D_{φ_j}cannot be bounded if $\varphi_j$has finite angular derivativeat any point on $\partial\mathbb{D}$ (see [2, Theorem 2.44]). Taking $w_j = 0$ in (1.2), we see that $\sum \left\|D_{\varphi_j}\right\| \geq \sqrt{\sum \frac{1+|\varphi_j(0)|^2}{\left(1-|\varphi_j(0)|^2\right)^3}}$,

which exhibits that $\sum \left\|D_{\varphi_j}\right\| \geq 1$ for each$\varphi_j$and that $\sum \left\|D_{\varphi_j}\right\| > 1$ whenever $\varphi_j(0) \neq 0$.

For the duration of this note, we will assume that $\varphi_j$is a nonconstant analytic map with$\sum\left\|\varphi_j\right\|_\infty < 1$. As mentioned earlier, this assumption guarantees that $D_{\varphi_j}$is compact on H^2.

2. **Adjoints, Norms, and Spectra**

We obtain information about the adjoint, norm, and spectrum of D_{φ_j}in certain specific cases. If $\varphi_j(z) = \frac{a^j z+b^j}{c^j z+d^j}$
is a no constant linear fractional self-map of $\mathbb{D}$, hence the map $\sigma_j(z) = \frac{\overline{a^J}z-\overline{c^J}}{-\overline{b^J}z+\overline{d^J}}$
also takes $\mathbb{D}$ into itself (see [1, Lemma 1]). It is not difficult to exhibit that $\sum\left\|\sigma_j\right\|_\infty <$ 1whenever $\sum\left\|\varphi_j\right\|_\infty < 1$. The link between these two maps has long beenregarded in reference to the adjoints of composition operators.

Theorem 1. For a pair of linear fractional maps $\varphi_j$and σ_j, as described above,

$$D^*_{\varphi_j}\left(T^j_{K^{(1)}_{\sigma_j(0)}}\right)^* = T^j_{K^{(1)}_{\varphi_j(0)}} D_{\sigma_j}$$

Proof. Observe that $K^{(1)}_{\varphi_j(0)}(z) = \frac{z}{\left(1-\left(\overline{b^J/d^J}\right)z\right)^2} = \frac{\overline{(d^J)}^2 z}{\left(-\overline{b^J}z+\overline{d^J}\right)^2}$

And $K^{(1)}_{\sigma_j(0)}(z) = \frac{z}{\left(1+\left(c^j/d^j\right)z\right)^2} = \frac{\left(d^j\right)^2 z}{\left(c^j z+d^j\right)^2}$.

Furthermore, see that; $T^j_{K^{(1)}_{\varphi_j(0)}} D_{\sigma_j}\left(K_{w_j}\right)(z) = T^j_{K^{(1)}_{\varphi_j(0)}} \frac{\overline{w_J}}{\left(1-\overline{w_J}\left(\frac{\overline{a^J}z-\overline{c^J}}{-\overline{b^J}z+\overline{d^J}}\right)\right)^2}$

UBJSR, Issue (39) Volume (2) December 2023- Arab Impact Factor, 2023: (AIF: 1.12)
Bakht Alruda University Quarterly Refereed Scientific Journal, ISSN: 1858-6139, Online
Received: 12/11/2023
Accepted:30/12/2023

$$= \frac{\overline{(d^j)^2} z}{\left(-\overline{b^j} z + \overline{d^j}\right)^2} \cdot \frac{\overline{w_j}}{\left(1 - \overline{w_j}\left(\frac{\overline{a^j} z - \overline{c^j}}{-\overline{b^j} z + \overline{d^j}}\right)\right)^2}$$
$$= \frac{\overline{(d^j)^2 w_j} z}{\left(-\overline{b^j} z + \overline{d^j} - \overline{a^j w_j} z + \overline{c^j w_j}\right)^2}$$
$$= \frac{\overline{(d^j)^2 w_j}}{\left(\overline{c^j w_j} + \overline{d^j}\right)^2} \cdot \frac{z}{\left(1 - \left(\frac{\overline{a^j w_j} + \overline{b^j}}{\overline{c^j w_j} + \overline{d^j}}\right) z\right)^2}$$

for any $w_j$in $\mathbb{D}$. Second hand,

$$D^*_{\varphi_j}\left(T^j_{K^{(1)}_{\sigma_j(0)}}\right)^*\left(K_{w_j}\right) = \frac{\overline{(d^j)^2 w_j}}{\left(\overline{c^j w_j} + \overline{d^j}\right)^2} D^*_{\varphi_j}\left(K_{w_j}\right) = \frac{\overline{(d^j)^2 w_j}}{\left(\overline{c^j w_j} + \overline{d^j}\right)^2} K^{(1)}_{\varphi_j(w_j)}.$$

Therefore, $D^*_{\varphi_j}\left(T^j_{K^{(1)}_{\sigma_j(0)}}\right)^*$ and $T^j_{K^{(1)}_{\varphi_j(0)}} D_{\sigma_j}$agree on the span of the reproducing kernelfunctions, which constitutes a dense subset of H^2. Therefore, the two operators are identical on H^2.

This result bears a close resemblance to Cowen's adjoint formula for composition operators (see [1, Theorem 2]), which can be rewritten $C^*_{\varphi_j}\left(T^j_{K^{(1)}_{\sigma_j(0)}}\right)^* = T^j_{K_{\varphi_j(0)}} C_{\sigma_j}$.

Let us focus on the specific situation where $\rho_j(z) = r_j z$for a real number r_j. In this case, Theorem 1 reduces to $D^*_{\rho_j}\left(T^j_z\right)^* = T^j_z D_{\rho_j}$. Therefore

$$D^*_{\rho_j} = D^*_{\rho_j}\left(T^j_z\right)^* T^j_z = T^j_z D_{\rho_j} T^j_z,$$

and therefore $D_{\rho_j} D^*_{\rho_j} = D_{\rho_j} T^j_z D_{\rho_j} T^j_z = \left(D_{\rho_j} T^j_z\right)^2$.

Note that; $\left(D_{\rho_j} T^j_z\right)^* = \left(T^j_z\right)^* D^*_{\rho_j} = \left(T^j_z\right)^* T^j_z D_{\rho_j} T^j_z = D_{\rho_j} T^j_z$,

so $D_{\rho_j} T^j_z$is self-adjoint. If$0 < r_j < 1$, we will see that $D_{\rho_j} T^j_z = \left|D^*_{\rho_j}\right|$. In specific,if we can determine the spectrum of $D_{\rho_j} T^j_z$, we will be able to compute $\left\|D_{\rho_j}\right\|$.

Theorem 2. If$\rho_j(z) = r_j z$for some real number $0 < r_j < 1$, hence

$$\sum \left\|D_{\rho_j}\right\| = \sum \left\lfloor \frac{1}{1 - r_j} \right\rfloor r_j^{\lfloor 1/(1 - r_j) \rfloor - 1}, \qquad 2.1$$

where $\lfloor\cdot\rfloor$denotes the greatest integer function.

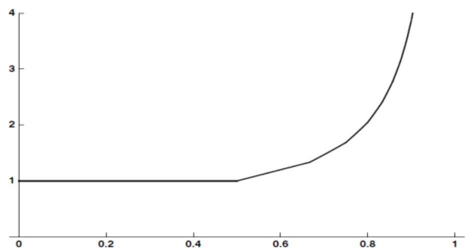

Figure 1. The quantity $\sum \left\|D_{\rho_j}\right\|$for different values of r_j.

Proof. Note that $D_{\rho_j} T^j_z(z^{n-1}) = D_{\rho_j}(z^n) = n\left(r_j z\right)^{n-1} = n r_j^{n-1} z^{n-1}$

UBJSR, Issue (39) Volume (2) December 2023- Arab Impact Factor, 2023: (AIF: 1.12)
Bakht Alruda University Quarterly Refereed Scientific Journal, ISSN: 1858-6139, Online
Received: 12/11/2023
Accepted:30/12/2023

for any natural number n. Therefore, the spectrum of $D_{\rho_j}T_z^j$ includes $\{nr_j^{n-1}: n \in \mathbb{N}\}$.
Second hand, suppose λ_j be an arbitrary eigenvalue for $D_{\rho_j}T_z^j$ with correspondingeigenvector f_j. Observe that; $\lambda_j f_j(z) = f_j(r_j z) + r_j z f_j'(r_j z)$.
If $f_j(0) \neq 0$, hence $\lambda_j = 1$. If $f_j(0) = 0$, differentiate both sides of the equation aboveto obtain $\lambda_j f_j'(z) = 2r_j f_j'(r_j z) + r_j^2 z f_j''(r_j z)$.
If $f_j'(0) \neq 0$, hence $\lambda_j = 2r_j$. Differentiating a total of $n - 1$ times, we obtain

$$\lambda_j f_j^{(n-1)}(z) = nr_j^{n-1} f_j^{(n-1)}(r_j z) + r_j^n z f_j^{(n)}(r_j z).$$

If $f_j^{(n-1)}(0) \neq 0$, Therefore $\lambda_j = nr_j^{n-1}$. Therefore, any eigenvalue must have the form nr_j^{n-1} for some natural number n. Since $D_{\rho_j}T_z^j$ is compact, its spectrum is exactly $\{0\} \cup \{nr_j^{n-1}: n \in \mathbb{N}\}$.
Since all the eigen values are positive, see that $\sum D_{\rho_j}T_z^j = \sum \left|D_{\rho_j}^*\right|$. Therefore,

$$\sum \left\|D_{\rho_j}\right\| = max\left\{\sum nr_j^{n-1} : n \in \mathbb{N}\right\}.$$

To obtain (2.1), observe that the function $h_j(x) = xr_j^{n-1}$ has precisely one local extremumon $[0, \infty)$, which are also its absolute maximum. Therefore we simply need to find the greatest natural number n so that $(n-1)r_j^{n-2} \leq nr_j^{n-1}$
or equivalently $1 - \frac{1}{n} \leq r_j$
or $n \leq \frac{1}{1-r_j}$.
Thus the quantity nr_j^{n-1} is maximized when $n = \lfloor 1/(1-r_j) \rfloor$.
There are several interesting consequences of Theorem 2. First of each, $\sum \left\|D_{\rho_j}\right\| = 1$ for $0 < r_j \leq 1/2$ and $\sum \left\|D_{\rho_j}\right\| > 1$ for $1/2 < r_j < 1$. Secondly, $\sum \left\|D_{\rho_j}\right\|$ tends to ∞ as r_j goes to 1. Since composition with a rotation is an isometry, (2.1) holds with r_j replaced by $|r_j|$ for any complex number r_j with $0 < \sum|r_j| < 1$. Likewise, the sameformula holds for $\sum \left\|D_{\varphi_j}\right\|$ where $\varphi_j(z) = r_j z^k$ for any k in $\mathbb{N}$.
We deduce this section with an observation relating to the spectrum of D_{ρ_j},which indeed pertains to a slightly larger class of operators.
Proposition 3. If $\varphi_j(z) = a^j z + b^j$, where $0 < \sum|a^j| < 1 - \sum|b^j|$, the operators D_{φ_j} are quasinilpotent; that is, its spectrum is $\{0\}$.
Proof. Assume that λ_j are a nonzero element of the spectrum of D_{φ_j}, which wouldnecessarily be an eigenvalue. Hence there is a nonzero function f_j in H^2 such that;

$$\lambda_j f_j(z) = f_j'(a^j z + b^j).$$

It is clear from this formula that f_j cannot be a polynomial. Differentiating n times,see that

$$\lambda_j f_j^{(n)}(z) = (a^j)^n f_j^{(n+1)}(a^j z + b^j),$$

so that $(\lambda_j/(a^j)^n) f_j^{(n)}(z) = (f_j^{(n)})'(a^j z + b^j)$.

UBJSR, Issue (39) Volume (2) December 2023- Arab Impact Factor, 2023: (AIF: 1.12)
Bakht Alruda University Quarterly Refereed Scientific Journal, ISSN: 1858-6139, Online
Received: 12/11/2023
Accepted:30/12/2023

Since ${f_j}^{(n)}$are not identically 0, we deduce that $\lambda_j/\left(a^j\right)^n$must be an eigenvalue for $D_{\varphi_j}$for every$n$, which is impossible. Therefore $\lambda_j = 0$are the only elements in the spectrumof D_{φ_j}.

Remark. Make a stronger statement than Proposition 3. When$\varphi_j(z) = a^j z + b^j$, note that $D_{\varphi_j}^n$are never equal to the trivial operators on H^2. Furthermore,for any natural number M, the closed linear span of the set$\{z^n\}_{n=0}^M$is invariantunder the operatorsD_{φ_j}. Therefore [3, Corollary 1] exhibits that $D_{\varphi_j}$are a compact universalquasinilpotent operators; that is, the norm closure of $\{WD_{\varphi_j}W^{-1}$: W is an invertible operator on $H^2\}$
contains every compact quasinilpotent operator.

3. Connections with composition operators

We can view the operatorsD_{φ_j}as the product $C_{\varphi_j}D$,although the benefit of this representation is limited by the fact that $D$is unboundedon H^2. The bounded operatorsD_{ρ_j}, where$\rho_j(z) = r_j z$for$0 < r_j < 1$, can act as asurrogate for Din the situation where$\sum\|\varphi_j\|_\infty < 1$. Recall that $D_{\rho_j}$are compact andquasinilpotent, and that we know both its adjoint and its norm.

Take $\sum\|\varphi_j\|_\infty \le \sum r_j < 1$ and define$(\varphi_j)_{r_j} = (1/r_j)\varphi_j$. Observe that $D_{\varphi_j} = C_{(\varphi_j)_{r_j}} D_{\rho_j}$.

(3.1)

Since $\sum \left\|D_{\varphi_j}\right\| = \sum \left\|C_{(\varphi_j)_{r_j}}\right\| \left\|D_{\rho_j}\right\|$, obtain the following estimate for $\left\|D_{\varphi_j}\right\|$.

Proposition 4. If $\varphi_j$are a nonconstant analytic self-map of $\mathbb{D}$ with $\sum\|\varphi_j\|_\infty < 1$, then

$$\sum \sqrt{\frac{1 + |\varphi_j(0)|^2}{\left(1 - |\varphi_j(0)|^2\right)^3}} \le \sum \left\|D_{\varphi_j}\right\| \le \sum \sqrt{\frac{r_j + |\varphi_j(0)|}{r_j - |\varphi_j(0)|}} \left\lfloor \frac{1}{1 - r_j} \right\rfloor r_j^{\lfloor 1/1 - r_j \rfloor - 1}$$

whenever $\sum\|\varphi_j\|_\infty \le \sum r_j < 1$.

If $\sum\|\varphi_j\|_\infty \le 1/2$ we may take $r_j = 1/2$to see that $\sum \sqrt{\frac{1+|\varphi_j(0)|^2}{\left(1-|\varphi_j(0)|^2\right)^3}} \le \sum \left\|D_{\varphi_j}\right\| \le \sum \sqrt{\frac{1+2|\varphi_j(0)|}{1-2|\varphi_j(0)|}}$.

In specific,$\sum \left\|D_{\varphi_j}\right\| = 1$ whenever both $\sum\|\varphi_j\|_\infty \le 1/2$and $\varphi_j(0) = 0$.

Example 5. Suppose$\varphi_j(z) = a^j z + b^j$, where $a^j \neq 0$and$\sum|a^j| + \sum|b^j| \le 1/2$, so that

$$\sum \sqrt{\frac{1+|b^j|^2}{\left(1-|b^j|^2\right)^3}} \le \sum \left\|D_{\varphi_j}\right\| \le \sum \sqrt{\frac{1+2|b^j|}{1-2|b^j|}}.$$

We can indeed improve on Proposition 4 because we know thenorm of $C_{(\varphi_j)_{r_j}}$precisely. For$r_j = 1/2$, it follows from [1, Theorem 3] that

$$\sum \left\|C_{(\varphi_j)_{r_j}}\right\| = \sum \sqrt{\frac{2}{1+4|a^j|^2-4|b^j|^2+\sqrt{\left(1-4|a^j|^2+4|b^j|^2\right)^2-16|b^j|^2}}}.$$

UBJSR, Issue (39) Volume (2) December 2023- Arab Impact Factor, 2023: (AIF: 1.12)
Bakht Alruda University Quarterly Refereed Scientific Journal, ISSN: 1858-6139, Online
Received: 12/11/2023
Accepted:30/12/2023

This value serves as a more precise upper bound for $\left\|D_{\varphi_j}\right\|$.
Working with D_{ρ_j}can also be useful for describing the adjoint of operatorsD_{φ_j}.
Collecting (3.1) with the identity $D^*_{\rho_j} = T^j_z D_{\rho_j} T^j_z$, we see that $D^*_{\varphi_j} = T^j_z D_{\rho_j} T^j_z C^*_{(\varphi_j)_{r_j}}$ whenever $\sum\left\|\varphi_j\right\|_\infty \leq \sum r_j < 1$. Thisanotationallows us to determine $D^*_{\varphi_j}$whenever we know $C^*_{(\varphi_j)_{r_j}}$.

Example 6. Suppose $\varphi_j(z) = r_j z^2$for $0 < r_j < 1$and peeking $(\varphi_j)_{r_j}(z) = z^2$. It is well
known (see [2, Exercise 9.1.1]) that $C^*_{(\varphi_j)_{r_j}}(f_j) = \frac{f_j(\sqrt{z}) + f_j(-\sqrt{z})}{2}$
for any $f_j$in H^2; that is, $C^*_{(\varphi_j)_{r_j}}(z^{2n}) = z^n$and $C^*_{(\varphi_j)_{r_j}}(z^{2n+1}) = 0$for any nonnegativeinteger n. Consequently, $D^*_{\varphi_j}(z^{2n}) = (n+1) r_j^n z^{n+1}$
and $D^*_{\varphi_j}(z^{2n+1}) = 0$. For example, note that

$$\sum D^*_{\varphi_j}\big(log(1-z)\big) = \sum D^*_{\varphi_j}\left(-\sum_{n=1}^{\infty}\frac{z^n}{n}\right) = -\sum\sum_{n=1}^{\infty}\frac{(n+1)r_j^n z^{n+1}}{2n}$$
$$= -\frac{z}{2}\sum\sum_{n=1}^{\infty}(r_j z)^n - \frac{z}{2}\sum\sum_{n=1}^{\infty}\frac{(r_j z)^n}{n}$$
$$= -\sum\frac{r_j z^2}{2(1-r_j z)} + \sum\frac{z\, log(1-r_j z)}{2}.$$

Observing that every $D_{\varphi_j}$from Example 6 has at least one nonzeroeigenvalue, since

$$D_{\varphi_j}(z^2) = 2(r_j z^2) = (2r_j)z^2.$$

Results:

1. Ohno [5] established a basic set of results relating to when the operators we are calling $D_{\varphi_j}$are bounded or compact on H^2.
2. Applying established results relating to composition operators, also obtain estimates for the norm of D_{φ_j}whenever $\sum\left\|\varphi_j\right\|_\infty < 1$ (Proposition 4).
3. This result bears a close resemblance to Cowen's adjoint formula for composition operators (see [1, Theorem 2]), which can be rewritten $C^*_{\varphi_j}\left(T^j_{K^{(1)}_{\sigma_j(0)}}\right)^* = T^j_{K_{\varphi_j(0)}} C_{\sigma_j}$.

Conclusion

In order to determine if the operators we are calling D_{φ_j} are bounded or compact on H^2, conclusion is quite similar to Cowen's adjoint formula for composition operators, which can be expressed as $C^*_{\varphi_j}\left(T^j_{K^{(1)}_{\sigma_j(0)}}\right)^* = T^j_{K_{\varphi_j(0)}} C_{\sigma_j}$.

UBJSR, Issue (39) Volume (2) December 2023- Arab Impact Factor, 2023: (AIF: 1.12)
Bakht Alruda University Quarterly Refereed Scientific Journal, ISSN: 1858-6139, Online
Received: 12/11/2023
Accepted:30/12/2023

References

[1] Carl C. Cowen, Linear fractional composition operators on H2, Integral Equations Operator Theory **11** (1988), no. 2, 151–160, DOI 10.1007/BF01272115. MR928479

[2] Carl C. Cowen and Barbara D. Mac Cluer, Composition operators on spaces of analytic functions, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1995. MR1397026

[3] Domingo A. Herrero, The Volterra operator is a compact universal quasinilpotent, Integral Equations Operator Theory **1** (1978), no. 4, 580–588, DOI 10.1007/BF01682942. MR516769

[4] R. A. Hibschweiler and N. Portnoy, Composition followed by differentiation between Bergman and Hardy spaces, Rocky Mountain J. Math. **35** (2005), no. 3, 843–855, DOI 10.1216/rmjm/1181069709. MR2150311

[5] Shuichi Ohno, Products of composition and differentiation between Hardy spaces, Bull. Austral.Math. Soc. **73** (2006), no. 2, 235–243, DOI 10.1017/S0004972700038818. MR2217942

[6] Dragan Vukotic, Analytic Toeplitz operators on the Hardy space Hp: a survey, Bull. Belg.Math. Soc. Simon Stevin **10** (2003), no. 1, 101–113. MR2032329Department of Mathematics, Shiraz Branch, Islamic Azad University, Shiraz, IranEmail address: fatehimahsa@yahoo.comDepartment of Mathematics and Statistics, Connecticut College,