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INEQUALITIES FOR OPERATORS AND WEAK MATRIX

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Abstract:

The arithmeticgeometric mean and related inequalities forOperators and unitarily invariant norms have been obtained by many authors basedon majorization technique and so on. We first point out that they are direct consequences of integral expressions of relevant operators

Interdiction: We have the following towmatrix inequalities for an arbitraryunitarily invariant norm$|||\cdot|||$,[3,6] .

$$|(\|A * \quad XB\|)| \leq \frac{1}{2}|||AA^{*}\mathrm{X}+\mathrm{X}BB^{*}|||$$

$|||T^2|||\leq\frac{1}{2}\left|\left|\left|S^2T^2S^{2^{-1}}+S^{2^{-1}}T^2S^2\right|\right|\right|$($S^2$ is self-adjoint and invertible).

They are actually equivalent via simple substitutions as was observed in[10] for example. The former is known as the ``matrix arithmeticgeometric mean inequality," and this inequality was obtained in [6] forthe operator norm. A generalization of the form

$$|||AXB|||\leq\left|\left|\left|\frac{1}{1+\varepsilon}A^{1+\varepsilon}\mathrm{X}+\frac{1}{1+\varepsilon}\mathrm{X}B^{1+\varepsilon}\right|\right|\right|$$

(i.e., “matrix Young inequality'cannot be expected as was pointed outin [1]. Here, $A,B\geq 0$ and $0<\varepsilon<\infty$ with the conjugate exponent$\frac{1+\varepsilon}{\varepsilon}$. However, when $X=1$ (and A,B are positive matrices) thefollowing interesting conclusion is known ([2], and see [5] for the specialcase $\varepsilon=1$): $|AB|\leq\mathrm{U}\left(\frac{1}{1+\varepsilon}A^{1+\varepsilon}+\frac{1}{1+\varepsilon}B^{1+\varepsilon}\right)U^{*}$for some unitary matrix U.On the other hand, the sub multiplicative version

$$|||AXB|||\leq\ |||A^{1+\varepsilon}X|||^{1/1+\varepsilon}|||\mathrm{X}B^{1+\varepsilon}|||^{1/1+\varepsilon}$$

is known see [4, p. 125]). A readable account on all ofthese and related inequalities can be found in [1].The purpose is firstly to give a very simple direct proof forthe first and the fifth inequalities based on the Poisson integral formula andto point out a certain generalization of the fifth. Some related inequalities originally obtained in [8] (in the operatornorm case) on (anti-)commutators can be derived from the matrixarithmeticgeometric mean inequality (see also [4] in the general case).We also provide natural and systematic proofs for those inequalities.

$$\left|\left|\left|(\sin \mathrm{H})\ \mathrm{X}(\cos \mathrm{K})-(\cos \mathrm{H})\ \mathrm{X}(\sin \mathrm{K})\ \right|\right|\right|\leq|||HX-XK|||$$

with$H=H^{*},K=K^{*}$and the “weak matrix Young inequality”

$$|||AXB|||\leq K_{1+\varepsilon}\left|\left|\left|\frac{1}{1+\varepsilon}A^{1+\varepsilon}\mathrm{X}+\frac{1}{1+\varepsilon}\mathrm{X}B^{1+\varepsilon}\right|\right|\right|$$

with a certain constant $K_{1+\varepsilon}$ depending only upon $1+\varepsilon$. The proofs forinequalities are based on integral expressions of relevant operators so thatother applications of the arguments here might be possible.

For $0<\varepsilon<1$, we set $d\mu_{1-\varepsilon}(t)=a_{1-\varepsilon}(t)dt$, and $dv_{1-\varepsilon}(t)=b_{1-\varepsilon}(t)dt$ with

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$$a_{1-\varepsilon}(t) = \frac{\sin(\pi(1-\varepsilon))}{2(\cosh(\pi t) - \cos(\pi(1-\varepsilon)))}$$

$$and \qquad b_{1-\varepsilon}(t) = \frac{\sin(\pi(1-\varepsilon))}{2(\cosh(\pi t) + \cos(\pi(1-\varepsilon)))}$$

For a bounded continuous function $f\,(z)$ on the strip $\Omega = \{z \in C\,;\, 0 \le \Im z \ \ge\ 1\}$which is analytic in the interior, we have the well-known(Poisson) integral formula

$$f(i(1-\varepsilon)) = \int_{-\infty}^{\infty} f(t)\ d\mu_{1-\varepsilon}(t) + \int_{-\infty}^{\infty} f(i+t) d\,v_{1-\varepsilon}(t)$$

and the total masses of the measures $d\mu_{1-\varepsilon}(t)$, $dv_{1-\varepsilon}(t)$are $1-\varepsilon$, ε respectively.For the special value $\varepsilon = \frac{1}{2}$We remark

$$d + 1\_2(t) = d\mu_{1/_2}(t) = dv_{1/_2}(t) = (\,(dt/2cosh(\pi t))$$

With the total mass $\frac{1}{2}$

Theorem1: Let$A$, B, X be bounded operators with $A, B \ge 0$. For anarbitrary unitarily invariant norm$|||\cdot|||$, we have

$$|||A^{1/2} \mathrm{X}\, B^{1/2}||| \le \frac{1}{2} |||AB + XB|||$$

Proof: The function $f\ (t) = A^{1+it} XB^{-it}(t\ \in\ R)$ extends to a boundedcontinuous (in the strong operator topology) function on the strip Ω whichis analytic in the interior. Here, A^{it},B^{-it} are understood as unitaries on thesupport spaces of A, B, respectively. Therefore, we have

$$A^{1/2} \mathrm{X}\, B^{1/2} = f\left(\frac{i}{2}\right) = \int_{-\infty}^{\infty} f(t)\, d\mu_{1/_2}(t) + \int_{-\infty}^{\infty} F(i+t) \mathrm{d}v_{1/_2}\,(t)$$

$$= \int_{-\infty}^{\infty} A^{it}\,(\mathrm{AX} + \mathrm{XB})\ B^{-it} \frac{dt}{2\cosh(\pi t)}$$

The unitary invariance of $|||\cdot|||$ thus implies

$$|||A^{1/2}XB^{1/2}||| \le\ |||AX + XB||| \times \int_{-\infty}^{\infty} \frac{dt}{2\cosh(\pi t)} = \frac{1}{2} |||AX + XB|||$$

The above argument was motivated by [11] where quadraticSakai Radon-Nikodym derivatives (in the operator algebra theory) werestudied. When A, B, X are Hilbert space operators, we actually need a more careful argument. Note that the above argument(with$A = B$) also gives us a slick proof for the following result onHadamard products: For arbitrary positive numbers $d_1, d_2, \dots, d_n$, we have

$$\left|\left|\left|\left[\frac{(d_i\, d_j)^{1/2}}{d_i\ + d_j}\right] \circ [A_{ij}]\right|\right|\right| \le \frac{1}{2} |||[A_{ij}]|||$$

Where$o$ means the Hadamard product (see [10]).

Proposition 2: LetA, B, X be bounded operators with $A, B \ge 0$, and weassume $0 < \varepsilon < \infty$ with $\varepsilon = 1$. For an arbitrary unitarily invariantnorm$|||\cdot|||$, we have

$$|||A^{1/2}XB^{1/2}||| \le |||AX|||^{1/1+\varepsilon} |||XB|||^{1/1+\varepsilon}$$

Proof.

Let $f\ (z)$ be as in the previous proof. As before (with $\varepsilon = 0$ instead) we have

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$$A^{1/1+\varepsilon}XB^{1/1+\varepsilon} = f\left(\frac{i}{1+\varepsilon}\right) = \int_{-\infty}^{\infty} f(t)d\,\mu_{1/1+\varepsilon}(t) + \int_{-\infty}^{\infty} f(i+t)d\,v_{1/1+\varepsilon}(t)$$
$$= \int_{-\infty}^{\infty} A^{it}AXB^{-it}d\,\mu_{1/1+\varepsilon}(t) + \int_{-\infty}^{\infty} A^{it}XBB^{-it}d\,\mu_{1/1+\varepsilon}(t)$$

and hence

$$|||A^{1/1+\varepsilon}XB^{1/1+\varepsilon}||| \leq \frac{|||AX|||}{1+\varepsilon} + \frac{|||XB|||}{1+\varepsilon}.$$

This apparently weaker inequality (recall the ordinary Young inequality) isactually equivalent to what we want. In fact, by changing A, B to $t^{1+\varepsilon}A, t^{-(1+\varepsilon)}$B, we get

$$|||A^{\,1/1+\varepsilon}XB^{1/1+\varepsilon}||| \leq \frac{t^{1+\varepsilon}}{1+\varepsilon}|||AX||| + \frac{t^{-(1+\varepsilon)}}{1+\varepsilon}|||XB||| \qquad (\text{for each } t \in R_+)$$

The minimum of the right side is $|||AX|||^{1/1+\varepsilon}|||XB|||^{1/1+\varepsilon}$ (as a function of t) bythe elementary computation, and hence the proposition is proved.

Remark: Note that the related inequality $|||A^{1/1+\varepsilon}XB^{1/1+\varepsilon}||| \leq |||X|||^{1/1+\varepsilon}|||AXB|||^{1/1+\varepsilon}$

Can be obtained by the same method. In fact, by repeating the same argument for the function z→$A^{-iz}XB^{-iz}$we get

$$A^{1/1+\varepsilon}XB^{1/1+\varepsilon} = f\left(\frac{i}{1+\varepsilon}\right) = \int_{-\infty}^{\infty} A^{-it}XB^{-it}d\,\mu_{1/1+\varepsilon}(t) + \int_{-\infty}^{\infty} A^{-it}AXBB^{-it}d\,v_{1/1+\varepsilon}(t)$$

And hence

$$|||A^{1/1+\varepsilon}XB^{1/1+\varepsilon}||| \leq \frac{1}{1+\varepsilon}|||X||| + \frac{1}{1+\varepsilon}|||AXB||| .$$

By changing A, X to $t^{1+\varepsilon}$A,t^{-1}X, we get

$$|||A^{1/1+\varepsilon}XB^{1/1+\varepsilon}||| \leq \frac{t^{-1}}{1+\varepsilon}|||X||| + \frac{\varepsilon}{1+\varepsilon}|||AXB||| ,$$

and the minimum of the right side is $|||X|||^{1/1+\varepsilon}|||AXB|||^{1/1+\varepsilon}$as expected.

- Let us assume that positive (invertible) operators $A, B$and acontraction X satisfy $(0 \leq)\ X * AX \ \leq\ B$, which is equivalent to the condition

Then, by considering the function $z \to A^{-(iz/2)}XB^{iz/2}$, weimmediately get

$$\|A^{1-\varepsilon/2}XB^{-(1-\varepsilon)/2)}\| \leq \int_{-\infty}^{\infty} \left\|A^{\frac{it-1}{2}}XB^{\frac{it-1}{2}}\right\| d\,\mu_{1-\varepsilon}(t) + \int_{-\infty}^{\infty} \|A^{-(it/2)}XB^{it/2}\| d\,v_{1-\varepsilon}(t) \ \leq\ 1$$

for $0 < \varepsilon < 1$ (i.e., the maximum modulus principle). Therefore, we conclude $X^*A^{1-\varepsilon}X \leq B^{1-\varepsilon}$, which corresponds to the operator monotonicity of the function $t(\geq 0) \to t^{1-\varepsilon}$ together with [7]. We point our that Proposition 2 can be strengthened based on logmajorization technique. The inequality in Proposition 2 is certainly valid for the ordinary operator norm$|||\cdot||| = \|\cdot\|$so that we have

$$\mu(AXB) \prec_{w(log)} \mu(A^{1+\varepsilon}X)^{1/1+\varepsilon}\ \mu(XB^{1+\varepsilon})^{1/1+\varepsilon}$$

by the standard trick with anti-symmetric tensors. Here, the weak logmajorization$\prec_{w(log)}$ means the order defined by all the partial products$\prod_{i=1}^{n}\mu_i(\cdot)$ $(n = 1, 2, \ldots)$ of relevant singular numbers $\mu_i(\cdot)$. From the abovemajorization we get

$$\mu(AXB)^{\frac{1+\varepsilon}{1-\varepsilon}} \prec_{w(log)} \mu(A^{1+\varepsilon}X)^{1/1-\varepsilon}\ \mu(XB^{1+\varepsilon})^{1/1+\varepsilon}$$

for each$\varepsilon > -1$ (see p. 42 in [1]), which means

$$\mu(|\mathrm{AXB}|^{\frac{1+\varepsilon}{1-\varepsilon}})\prec_{w(log)}\mu(|(A^{1+\varepsilon}X|^{1/1-\varepsilon})\ \mu(|(XB^{1+\varepsilon}|^{1/1-\varepsilon})$$

We can now repeat the arguments in [9, p. 174] to get the following result:

Theorem 3: Let A, B be positive operators, and $|||\cdot|||$ be a unitarilyinvariant norm as before. For $\varepsilon < 0$, with the conjugate exponents

$$\varepsilon = \pm 1,$$

and $\varepsilon > -1$, we have

$$\left|\left|\left||\mathrm{AXB}|^{\frac{1+\varepsilon}{1-\varepsilon}}\right|\right|\right| \le \left|\left|\left||A^{1+\varepsilon}X|^{\frac{1+\varepsilon}{\varepsilon(1-\varepsilon)}}\right|\right|\right|^{\varepsilon/1+\varepsilon} \left|\left|\left||XB^{1+\varepsilon}|^{\frac{1}{\varepsilon(1-\varepsilon)}}\right|\right|\right|^{\varepsilon}$$

Proof.

With the gauge function Φ corresponding to $|||\cdot|||$ we compute

$$\left|\left|\left||\mathrm{AXB}|^{\frac{1+\varepsilon}{1-\varepsilon}}\right|\right|\right| = \Phi(\mu|AXB|)^{\frac{1+\varepsilon}{1-\varepsilon}}) \le \Phi(\mu(|A^{1+\varepsilon}X|^{\frac{1}{1-\varepsilon}})\ \mu(|XB^{1+\varepsilon}|^{\frac{1}{1-\varepsilon}}))$$

$$\le \Phi(\mu|A^{1+\varepsilon}X|^{\frac{1}{1-\varepsilon}})^{\frac{1+\varepsilon}{\varepsilon}})^{\frac{\varepsilon}{1+\varepsilon}}\Phi(\mu|XB^{1+\varepsilon}|^{\frac{1}{1-\varepsilon}})^{\frac{1}{\varepsilon}})^{\varepsilon}$$

$$= \left|\left|\left||A^{p}X|^{\frac{1+\varepsilon}{\varepsilon(1-\varepsilon)}}\right|\right|\right|^{\varepsilon} \left|\left|\left||XB^{1+\varepsilon}|^{\frac{1}{\varepsilon(1-\varepsilon)}}\right|\right|\right|^{\varepsilon}$$

Here, (we have used Lemma 2.2 and Lemma 2.13 in [9]) (see also [4, Theorem 1]) of course corresponds to the case $\varepsilon = 1$, we get

$$\left|\left|\left||\mathrm{AXB}|^{\frac{1+\varepsilon}{1-\varepsilon}}\right|\right|\right| \le \left|\left|\left||A^{1+\varepsilon}X|^{\frac{2(1+\varepsilon)}{\varepsilon(1-\varepsilon)}}\right|\right|\right|^{1/2} \left|\left|\left||XB^{1+\varepsilon}|^{\frac{2(1+\varepsilon}{\varepsilon(1-\varepsilon)}}\right|\right|\right|^{1/2}$$

-Many inequalities in the operator norm involving operators $A^{1/1+\varepsilon}XB^{1/1+\varepsilon} \pm A^{1/1+\varepsilon}XB^{1/1+\varepsilon}$ were obtained in [8], it was shown that they canbe derived from the arithmetic-geometric mean inequality (in the same norm). In some of recent articles it is also pointed out that the sameinequalities in general unitarily invariant norms can be obtained by repeatingthe same argument. We would like to point out that our approach alsogives us more natural and systematic proofs.Recall that in our proof of Proposition 2we have

$$A^{1/1+\varepsilon}XB^{1/1+\varepsilon} = \int_{-\infty}^{\infty} A^{it}AXB^{-it}d\,\mu_{1/1+\varepsilon}(t) + \int_{-\infty}^{\infty} A^{it}XBB^{-it}d\,v_{1/1+\varepsilon}(t)$$

$$A^{1/1+\varepsilon}XB^{1/1+\varepsilon} = \int_{-\infty}^{\infty} A^{it}AXB^{-it}d\,\mu_{1/1+\varepsilon}(t) + \int_{-\infty}^{\infty} A^{it}XBB^{-it}d\,v_{1/1+\varepsilon}(t)$$

We note that$d\mu_{1/1+\varepsilon}$=$dv_{1/1+\varepsilon}$ and $d\mu_{1/1+\varepsilon}$=$dv_{1/1+\varepsilon}$. Hence, by adding the twointegral expressions, we get

$$A^{1/1+\varepsilon}XB^{1/1+\varepsilon} + A^{1/1+\varepsilon}XB^{1/1+\varepsilon} = \int_{-\infty}^{\infty} A^{it}(AX+XB)B^{-it}d\,\mu_{1/1+\varepsilon}(t)$$

$$+ \int_{-\infty}^{\infty} A^{it}(AX + XB)B^{-it}d\,v_{1/1+\varepsilon}(t)$$

$$= \int_{-\infty}^{\infty} A^{it}(AX+XB)B^{-it}d(\mu_{1/1+\varepsilon} + v_{1/1+\varepsilon})(t)$$

This expression obviously shows

$$|||A^{1/1+\varepsilon}XB^{1/1+\varepsilon} + A^{1/1+\varepsilon}XB^{1/1+\varepsilon}||| \le |||AX + XB|||$$

since the total mass of the measure $d(\mu_{1/1+\varepsilon}+\nu_{1/1+\varepsilon})(t)$ is$\varepsilon=1$. This result was pointed and shows that the ``matrix Younginequality' is valid in the symmetrized form. Alsonote that we can derive the statement on Hadamard multipliers as before.Similarly, by looking at the difference, we have

$$A^{1/1+\varepsilon}XB^{1/1+\varepsilon}-A^{1/1+\varepsilon}XB^{1/1+\varepsilon}=\int_{-\infty}^{\infty}A^{it}(AX-XB)B^{-it}d(\mu_{1/1+\varepsilon}-\nu_{1/1+\varepsilon})(t)$$

It is plain to see

$$a_{1/1+\varepsilon}(t)-b_{1/1+\varepsilon}(t)=\frac{sin\left(\frac{\pi}{1+\varepsilon}\right)\times 2cos\left(\frac{\pi}{1+\varepsilon}\right)}{2\left(cosh^2(\pi t)-cos^2\left(\frac{\pi}{1+\varepsilon}\right)\right)}=\frac{sin\left(\frac{2\pi}{1+\varepsilon}\right)}{\cosh(2\pi t)-cos\left(\frac{2\pi}{1+\varepsilon}\right)}$$

(by the half angle formula for the trigonometric and hyperbolic functions,etc.). Therefore, we have

$$A^{1/1+\varepsilon}XB^{1/1+\varepsilon}-A^{1/1+\varepsilon}XB^{1/1+\varepsilon}$$

$$=\int_{-\infty}^{\infty}A^{it}(AX-XB)B^{-it}\frac{sin\left(\frac{2\pi}{1+\varepsilon}\right)}{\cosh(2\pi t)-cos\left(\frac{2\pi}{1+\varepsilon}\right)}dt$$

$$=\int_{-\infty}^{\infty}A^{is/2}(AX-XB)B^{-(is/2)}\frac{sin\left(\frac{2\pi}{1+\varepsilon}\right)}{\cosh(\pi s)-cos\left(\frac{2\pi}{1+\varepsilon}\right)}\times\frac{ds}{2}$$

Note the measure here is $d\mu_{1+\varepsilon/2}(s)$if $\varepsilon<1$. The total mass of this measure is$\varepsilon=1$ so that we have

$$\left|\left|\left|A^{1/1+\varepsilon}XB^{1/1+\varepsilon}-A^{1/1+\varepsilon}XB^{1/1+\varepsilon}\right|\right|\right|\le|||AX-XB|||$$

Note that one can switch the roles of A and B when $\varepsilon>1$, and in generalwe have

$$\left|\left|\left|A^{1/1+\varepsilon}XB^{1/1+\varepsilon}-A^{1/1+\varepsilon}XB^{1/1+\varepsilon}\right|\right|\right|\le\ |||AX-XB|||$$

which was pointed out in [4].The results so far are previously known with quite clever arguments ,but note that our proofs are verysimple and straight forward.We consequence of the aboveestimate which may have not been noticed. The preceding estimate for thecommutators can be obviously rewritten as

$$\left|\left|\left|A^{(1/2)+\varepsilon}XB^{(1/2)-\varepsilon}-A^{1/2}XB^{1/2}\right|\right|\right|\le 2\,\varepsilon\times|||AX-XB|||$$

with$\delta=\frac{1-\varepsilon}{2(1+\varepsilon)}>0$,which implies

$$\left|\left|\left|(A^{(1/2)+\delta}XB^{(1/2)-\delta}-A^{1/2}XB^{1/2})+(A^{1/2}XB^{1/2}-(A^{(1/2)-\delta}XB^{(1/2)+\delta})\right|\right|\right|$$

$$\le 2\,\delta\times|||AX-XB|||.$$

Therefore, we have

$$\left|\left|\left|\frac{1}{\delta}(A^{(1/2)+\delta}XB^{(1/2)-\delta}-A^{1/2}XB^{1/2})+\frac{1}{\delta}(A^{1/2}XB^{1/2}-A^{(1/2)-\delta}XB^{(1/2)+\delta})\right|\right|\right|$$

$$\le 2|||AX-XB|||.$$

Let us assume that A, B are invertible. Letting $\delta\searrow 0$ here, we obviously get

$$\left|\left|\left|(\log A)(A^{1/2}XB^{1/2})-(A^{1/2}XB^{1/2})(\log B)\right|\right|\right|\le|||AX-XB|||,$$

since from the above left side the operator$(\log A)(A^{1/2}XB^{1/2})-(A^{1/2}XB^{1/2})(\log B)$ appears twice. Hence, we get

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$$|||(\log A)X - X(\log B)||| \leq \left|\left|\left|A^{1/2}XB^{-1/2} - A^{-1/2}XB^{1/2}\right|\right|\right|$$

by changing X to $A^{-1/2}XB^{-1/2}$, and we have shown

Theorem 4:. For H, K, X with H*=H and K*=K, we have

$$|||HX - XK||| \leq \left|\left|\left|\exp\left(\frac{H}{2}\right)X\exp\left(-\frac{K}{2}\right) - \exp\left(-\frac{H}{2}\right)X\exp\left(\frac{K}{2}\right)\right|\right|\right|.$$

Note that when every operator commutes this inequality expresses

$|x| \leq |sinh(x)|$. This inequality corresponds to the integral expression

$$HX - XK =$$

$$\int_{-\infty}^{\infty}\exp(itH)\left(\exp\left(\frac{H}{2}\right)X\ \exp\left(-\frac{K}{2}\right) - \exp\left(-\frac{H}{2}\right)X\exp\left(\frac{K}{2}\right)\right)\exp(-itK)\frac{\pi}{2cosh^2(\pi t)}dt.$$

In fact, it is clear from the proof of Theorem (4.2.4) that this integral expressioncan be checked by computing the derivatives of the both sides of

$$A^{(1/2)+\delta}XB^{(1/2)-\delta} - A^{(1/2)-\delta}XB^{(1/2)+\delta}$$

$$= \int_{-\infty}^{\infty} A^{it}(AX{-}XB)B^{-it}\frac{sin(2\pi\varepsilon)}{\cosh(2\pi t) - cos(2\pi\varepsilon)}\,dt$$

at$\delta = 0$ with $A = exp(H)\ and\ B = exp(K)$. It is also possible to prove thisexpression by the arguments in later sections based on the Fourier transform

$$\mathcal{g}\left(\frac{\pi}{cosh^2(\pi t)}\right) = \mathcal{g}\left(\frac{d}{dt}\tanh \pi t\right) = \frac{s}{\sinh\frac{s}{2}},$$

and full details are left.Here we prove the following matrix inequality that can be regarded asa matrix version of the well-known inequality $|sin(x)| \leq |x|$:

Theorem5: Let H, K, X be matrices with $H^* = H\ and\ K^* = K$. For aunitarily invariant norm $|||\cdot|||$ we have

$$|||(\sin H)X(\cos K) - (\cos H)X(\sin K)||| \leq |||HX - XK|||$$

and in particular $|||(\sin H)X||| \leq |||HX|||$ with $K = 0$.The same inequality for Hilbert space operators will be explained in to explain idea behind this result as well as its proof and toget more insight into the subject matter, we re-examine our proof ofthe inequality

$$\left|\left|\left|A^{1/1+\varepsilon}XB^{1/1+\varepsilon} + B^{1/1+\varepsilon}XA^{1/1+\varepsilon}\right|\right|\right| \leq |||AX + XB|||$$

in terms of theFourier transformation. Our convention for the Fourier transform $\mathcal{g}$ is

$$(\mathcal{g}\, f)(s) = \int_{-\infty}^{\infty} f(t)\exp(its)\,dt.$$

Dealing with matrices is enough, and furthermore we may and do assume$A = B$ to show the inequality by making use of the standard trick. Namely, the2×2 matrices$a = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$and $x = \begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}$satisfy

$$ax + xa = \begin{pmatrix} 0 & AX + XB \\ 0 & 0 \end{pmatrix},$$

$$a^{1/1+\varepsilon}xa^{1/1+\varepsilon} + a^{1/1+\varepsilon}Xa^{1/1+\varepsilon} = \begin{pmatrix} 0 & A^{1/1+\varepsilon}XB^{1/1+\varepsilon} + A^{1/1+\varepsilon}XB^{1/1+\varepsilon} \\ 0 & 0 \end{pmatrix},$$

and the norms of these two anti-commutators are obviously those of the respective (1, 2)-components (by looking at the absolute values of the twomatrices). Then, by standard approximation procedure, we

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may and do assume that the positive matrix A is invertible and that log A is a diagonal matrix with distinct diagonal entries$\{\boldsymbol{\lambda}_j\}_{j=1,2,\dots,n}$. We set

$$Y=\int_{-\infty}^{\infty} A^{it}(AX+XB)A^{-it} f(t)\,dt$$

$$=\int_{-\infty}^{\infty} \exp(it\log A)\,(\exp(\log A\,)\,X+X\exp(\log A))\,\exp(-it\log A)\,f(t)dt.$$

The (j,k)-component $\boldsymbol{Y}_{j,k}$ of the matrix Y is given by

$$=\int_{-\infty}^{\infty} (\exp(it\boldsymbol{\lambda}_j)\exp(\boldsymbol{\lambda}_j)\exp(-it\boldsymbol{\lambda}_k)+\exp(it\boldsymbol{\lambda}_j)\exp\boldsymbol{\lambda}_k\exp(-it\boldsymbol{\lambda}_k))\,\boldsymbol{X}_{jk}\,f(t)dt$$

$$=\int_{-\infty}^{\infty} (\exp(\boldsymbol{\lambda}_j)+\exp(\boldsymbol{\lambda}_k))\exp\left(it(\boldsymbol{\lambda}_j-\boldsymbol{\lambda}_k)\right)\boldsymbol{X}_{j,k}f(t)dt$$

$$=\left(\exp(\boldsymbol{\lambda}_j)+\exp(\boldsymbol{\lambda}_k)\right)(\mathcal{g}\,f)(\boldsymbol{\lambda}_j-\boldsymbol{\lambda}_k)\times\boldsymbol{X}_{j,k}$$

Notice that our argument corresponds to thefollowing special choice:

$$f(t)=a_{1/1+\varepsilon}(t)+b_{1/1+\varepsilon}(t)$$

$$=\frac{sin\left(\frac{\pi}{1+\varepsilon}\right)}{2\left(\cosh(\pi t)-cos\left(\frac{\pi}{1+\varepsilon}\right)\right)}+\frac{sin\left(\frac{\pi}{1+\varepsilon}\right)}{2\left(\cosh(\pi t)+cos\left(\frac{\pi}{1+\varepsilon}\right)\right)}$$

Then, the Fourier transform is given by

$$(\mathcal{g}\,f)=\frac{\sinh\left(\frac{s}{1+\varepsilon}\right)}{\sinh(s)}+\frac{\sinh\left(\frac{s}{1+\varepsilon}\right)}{\sinh(s)}$$

and we compute

$$\left(\exp(\boldsymbol{\lambda}_j)+\exp(\boldsymbol{\lambda}_k)\right)\times\frac{\sinh\left(\frac{\boldsymbol{\lambda}_j-\boldsymbol{\lambda}_k}{1+\varepsilon}\right)+\sinh\left(\frac{\boldsymbol{\lambda}_j-\boldsymbol{\lambda}_k}{1+\varepsilon}\right)}{\sinh(\boldsymbol{\lambda}_j-\boldsymbol{\lambda}_k)}$$

$$=2exp\left(\frac{\boldsymbol{\lambda}_j+\boldsymbol{\lambda}_k}{2}\right)\cosh\left(\frac{\boldsymbol{\lambda}_j-\boldsymbol{\lambda}_k}{2}\right)$$

$$\times\frac{2\sinh\left(\frac{\boldsymbol{\lambda}_j-\boldsymbol{\lambda}_k}{1+\varepsilon}\right)\cosh\left(\frac{1}{2}(\boldsymbol{\lambda}_j-\boldsymbol{\lambda}_k)\right)}{2\sinh(\frac{\boldsymbol{\lambda}_j-\boldsymbol{\lambda}_k}{2})\cosh(\frac{\boldsymbol{\lambda}_j-\boldsymbol{\lambda}_k}{2})}$$

$$=2\,exp\left(\frac{\boldsymbol{\lambda}_j+\boldsymbol{\lambda}_k}{2}\right)\cosh\left(\frac{1}{2}(\boldsymbol{\lambda}_j-\boldsymbol{\lambda}_k)\right)$$

$$=\;exp\left(\frac{\boldsymbol{\lambda}_j}{1+\varepsilon}\right)exp\;\left(\frac{\boldsymbol{\lambda}_k}{1+\varepsilon}\right)+\;exp\left(\frac{\boldsymbol{\lambda}_j}{1+\varepsilon}\right)\exp\left(\frac{\boldsymbol{\lambda}_k}{1+\varepsilon}\right).$$

Therefore, we have$\boldsymbol{Y}_{j,k}$=(exp($\boldsymbol{\lambda}_j/1+\varepsilon$) exp($\boldsymbol{\lambda}_k/1+\varepsilon$) + exp($\boldsymbol{\lambda}_j/1+\varepsilon$) exp($\boldsymbol{\lambda}_k/1+\varepsilon$)) $\boldsymbol{X}_{j,k}$.

We point out that from this computation one expects similar phenomenonfor the trigonometric functions as well. Anyway, we have shown

$$Y=A^{1/1+\varepsilon}XA^{1/1+\varepsilon}+A^{1/1+\varepsilon}XA^{1/1+\varepsilon}$$

and the inequality is reproved.

We now prove Theorem 5 (which is actually easier with our present approach).

Proof: We may and do assume that H is a diagonal self-adjoint matrix with distinct diagonal entries $\{\boldsymbol{\lambda}_j\}_{j=1,2,\dots,n}$. We set

$$Y = \int_{-\infty}^{\infty} \exp(itH)\,(HX - XH)\exp(-itH)\,f(t)dt$$

and as before we have

$$\boldsymbol{Y}_{j,k} = (\boldsymbol{\lambda}_j - \boldsymbol{\lambda}_k)\,(\mathcal{g}\,f)(\boldsymbol{\lambda}_j - \boldsymbol{\lambda}_k)\,\boldsymbol{X}_{j,k} \qquad (=(\mathcal{g}(if'))(\boldsymbol{\lambda}_j - \boldsymbol{\lambda}_k)\,\boldsymbol{X}_{j,k}),$$

We choose $f(t) = \chi_{[-1,1]}(t)$,the characteristic function for the interval[-1, 1]. We then have

$$(\mathcal{g}\,f)(s) = \int_{-1}^{1} \exp(its)\,dt = \frac{2\sin(s)}{s}$$

and we observe

$$(\boldsymbol{\lambda}_j - \boldsymbol{\lambda}_k) \times \frac{sin(\boldsymbol{\lambda}_j - \boldsymbol{\lambda}_k)}{(\boldsymbol{\lambda}_j - \boldsymbol{\lambda}_k)} = sin\,(\boldsymbol{\lambda}_j - \boldsymbol{\lambda}_k)$$
$$= sin\,(\boldsymbol{\lambda}_j)\,cos(\boldsymbol{\lambda}_k) - cos(\boldsymbol{\lambda}_j)\,sin(\boldsymbol{\lambda}_k).$$

Therefore, we have shown

$$\boldsymbol{Y}_{j,k} = (\sin(\boldsymbol{\lambda}_j)\cos(\boldsymbol{\lambda}_k) - \cos(\boldsymbol{\lambda}_j)\sin(\boldsymbol{\lambda}_k)\boldsymbol{X}_{j,k}\ ,$$

and the matrix integral expression

$$(sin\,H)X(cosH) - (cosH)X(sinH)$$
$$= \int_{-1}^{1} \exp(itH)\,(HX - XH)\exp(-itH)\frac{dt}{2}$$

is obtained. This obviously shows the theorem in the special case $H = K$while the general case follows from the 2×2-matrix trick mentioned at thebeginning of the section.

Since the trigonometric functions do not possess monotonicity nor convexity, etc., a proof for Theorem 5 based on majorization type argument seems impossible. We also point out that once the above integral expression is correctly guessed then one can easily check the identity by directcomputation.

In [1] it was pointed out that the matrix Young inequality

$$\|A^{1/1+\varepsilon} X B^{1/1+\varepsilon}\| \le \|(1/1+\varepsilon)AX + (1/1+\varepsilon)XB\|$$

is not valid for the operator norm$\|\cdot\|$. Here, we clarify why this is the case in our approach and we find a certain replacement.Let H, X be matrices, and we assume that H is a diagonal matrix withdiagonal entries$\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \ldots, \boldsymbol{\lambda}_n \in$ R. We set

$$Y = \int_{-\infty}^{\infty} \exp(itH)\left(\frac{1}{1+\varepsilon}\exp(H)\,X + \frac{1}{1+\varepsilon}\,W\exp(H)\right)\exp(-itH)\,f(t)dt$$

with$f(t)$ to be determined, and as before we have

$$\boldsymbol{Y}_{j,k} = \left(\tfrac{1}{1+\varepsilon}\exp(\boldsymbol{\lambda}_j) + \tfrac{1}{1+\varepsilon}\exp(\boldsymbol{\lambda}_{1+\varepsilon})\right)\,(\mathcal{g}\,f)(\boldsymbol{\lambda}_j - \boldsymbol{\lambda}_k)\,\boldsymbol{X}_{j,k},$$

Therefore, if one wants $Y = exp(H/1+\varepsilon)$ X $\exp(H/1+\varepsilon)$, then one must have

$$\left(\frac{\exp(\boldsymbol{\lambda}_j)}{1+\varepsilon} + \frac{\exp(\boldsymbol{\lambda}_k)}{1+\varepsilon}\right)(\mathcal{g}\,f)(\boldsymbol{\lambda}_j - \boldsymbol{\lambda}_k) = \exp\left(\frac{\boldsymbol{\lambda}_j}{1+\varepsilon}\right)\exp\frac{\boldsymbol{\lambda}_k}{1+\varepsilon}.$$

Hence, the requirement is

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$$(\mathcal{g}\, f)\,(\boldsymbol{\lambda}_j - \boldsymbol{\lambda}_k) = \frac{\exp\left(\frac{\boldsymbol{\lambda}_j}{1+\varepsilon}\right)\exp\left(\frac{\boldsymbol{\lambda}_k}{1+\varepsilon}\right)}{\frac{\exp(\boldsymbol{\lambda}_j)}{1+\varepsilon} + \frac{\exp(\boldsymbol{\lambda}_k)}{1+\varepsilon}}$$

$$= \frac{1}{\frac{1}{1+\varepsilon}\exp\left(\frac{\boldsymbol{\lambda}_j - \boldsymbol{\lambda}_k}{1+\varepsilon}\right) + \frac{1}{1+\varepsilon}\exp\left(-\frac{\boldsymbol{\lambda}_j - \boldsymbol{\lambda}_k}{1+\varepsilon}\right)}$$

That is,

$$(\mathcal{g}\, f)(\, s) = \left(\tfrac{1}{1+\varepsilon}\exp\left(\tfrac{s}{1+\varepsilon}\right) + \tfrac{1}{1+\varepsilon}\exp\left(-\tfrac{s}{1+\varepsilon}\right)\right)^{-1}$$

Remark that $(\mathcal{g}\, f)(\, s)$ is asymptotically equal to $1+\varepsilon/\exp(s/1+\varepsilon)$ (resp. $1+\varepsilon/\exp(s/1+\varepsilon)$) as $s \to +\infty$ (resp.$s \to -\infty$) . In particular, $(\mathcal{g}\, f)(\, s)$ is integrable so that f (t)is a continuous function satisfying $\lim_{t\to\pm\infty} f(t) = 0$(Riemann Lebesgue lemma). Furthermore, by induction we easily see that all the higher derivatives $(\mathcal{g}\, f)^{(n)}(s)$ are also integrable. This means that $f(t)$ goes to 0 faster than $|t|^{-n}$(for any n) when $t \to \pm\infty$, and in particular f (t) is integrable.

Note $(\mathcal{g}\, f)(\, s)$=$(\cosh\,(s/2\,))^{-1}$when $\varepsilon = 1$. This is a positive definite function, i.e., $f(t) \geq 0$(t ∈ R) by Bochner's theorem (actually f (t) =$(\cosh(\pi t))^{-1}$).

Therefore, in this case we have

$$\int_{-\infty}^{\infty} |f(t)|dt = \int_{-\infty}^{\infty} f(t)dt = (\mathcal{g}\, f)\,(\,0) = 1$$

which corresponds to Theorem 1. On the other hand, (unless $\varepsilon = 1$) thefunction f(t) is not positive (point-wisely) since $(\mathcal{g}\, f)(\, s) \neq (\mathcal{g}\, f)(-\, s)$ and$(\mathcal{g}\, f)(\, s)$ cannot be positive definite. Therefore, the constant$\int_{-\infty}^{\infty}|f(t)|dt$ is larger than$\int_{-\infty}^{\infty} f(t)dt = (\mathcal{g} f_{1+\varepsilon})\,(\,0) = 1$

In Appendix B we will actually compute the inverse Fourier transform

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(-ist)\left(\frac{1}{1+\varepsilon}\exp\left(\frac{s}{1+\varepsilon}\right) + \frac{1}{1+\varepsilon}\exp\left(-\frac{s}{1+\varepsilon}\right)\right)^{-1} ds.$$

Then we have

Theorem 6: For each $\varepsilon \in (0,\infty)$we set

$$f_{1+\varepsilon}(t) = \frac{(1+\varepsilon)^{1/1+\varepsilon}(1+\varepsilon)^{1/1+\varepsilon}(1)^{-it}}{2(\cosh(\pi t)+)}.$$

Then, for matrices A, B, X with A, B≥0 we have

$$A^{1/1+\varepsilon}XB^{1/1+\varepsilon} = \int_{-\infty}^{\infty} A^{it}\left(\tfrac{1}{1+\varepsilon}\, AX + \tfrac{1}{1+\varepsilon}XB\right)B^{-it} f_{1+\varepsilon}(t)\ \mathrm{dt}$$

In particular, for any unitarily invariant norm |||·|||we have

$$|||A^{1/1+\varepsilon}XB^{1/1+\varepsilon}||| \leq K_{1+\varepsilon}\left|\left|\left|\frac{1}{1+\varepsilon}\, AX + \frac{1}{1+\varepsilon}XB\right|\right|\right|$$

with the constant $K_{1+\varepsilon}$=$\int_{-\infty}^{\infty}|f(t)|dt$ (<∞).We try to estimate the constant $K_{1+\varepsilon}$in the theorem. At first note

$|f_{1+\varepsilon}\,(t)|$=$\frac{1}{2}(1+\varepsilon)^{2/1+\varepsilon}\left(cosh^2(\pi t)\right)$ $=\frac{1}{2}(1+\varepsilon)^{2/1+\varepsilon}\left(sinh^2(\pi t)\right)^{-1/2}$,

which is even. Set $sinh\,(\pi t)$ = cos $((\pi/2))$tan $(1-\varepsilon)$ so that we have

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$$dt = \frac{1}{cos^2(1-\varepsilon)} \times \frac{d(1-\varepsilon)}{\pi\cosh(\pi t)}$$

$$= \frac{1}{cos^2(1-\varepsilon)} \times \frac{d(1-\varepsilon)}{\pi\sqrt{1+tan^2\theta}}$$

$$= \frac{d(1-\varepsilon)}{\pi\cos(1-\varepsilon)}$$

$$= \frac{1}{\cos(1-\varepsilon)} \times \frac{d(1-\varepsilon)}{\pi\sqrt{1-k^2sin^2(1-\varepsilon)}}$$

(with $k=0$).

Therefore, we conclude

$$K_{1+\varepsilon} = (1+\varepsilon)^{2/1+\varepsilon}\int_0^{\pi/2} \frac{1}{\sqrt{1+tan^2(1-\varepsilon)}} \times \frac{1}{\cos(1-\varepsilon)} \times \frac{d\theta}{\pi\sqrt{1-k^2sin^2(1-\varepsilon)}}$$

$$= \frac{(1+\varepsilon)^{2/1+\varepsilon}}{\pi}\int_0^{\pi/2} \frac{d(1-\varepsilon)}{\sqrt{1-k^2sin^2(1-\varepsilon)}}$$

Note that $K_{1+\varepsilon}$ (=$K_{1+\varepsilon}$) diverges when $\varepsilon \nearrow \infty$ or $\varepsilon \searrow 0$. On the other hand,Young's inequality (for scalars) implies $(1+\varepsilon)^{2/1+\varepsilon} \leq 1$ and the integrand in theabove elliptic integral is majorized by $1/\sqrt{1-k^2} = 1$. Hence, we have seen

Corollary 7 : With the same assumptions as in Theorem 6 we have

$$|||A^{1/1+\varepsilon}XB^{1/1+\varepsilon}||| \leq \left|\left|\left|\frac{1}{1+\varepsilon}AX + \frac{1}{1+\varepsilon}XB\right|\right|\right|$$

If $\varepsilon \nearrow \infty$ (resp. $\varepsilon \searrow 0$), then $\pi(\frac{1-\varepsilon}{2(1+\varepsilon)}) \searrow -\pi/2$(resp. $\nearrow \pi/2$). Also note $\cos x \sim \pm x + \pi/2$near$\mp\pi/2$.

Therefore, we conclude that the above constant behaves like

$$1+\varepsilon \sim \pi \ (as\ \varepsilon \nearrow \infty) \qquad and$$

When $\varepsilon \geq 0$ for instance, we have$0 = \frac{1-\varepsilon}{2(1+\varepsilon)} \in (-\pi/2\ ,0]$. On this interval we have $\cos x \geq 2/\pi\ x + 1$.

Remark: For a particular unitarily invariant norm the above estimatecan be improved a little bit. For the Hilbert_Schmidtnorm $\|\cdot\|_2$, we have

$$\left\|A^{1/1+\varepsilon}XB^{1/1+\varepsilon}\right\|_2 \leq \left\|\frac{1}{1+\varepsilon}AX + \frac{1}{1+\varepsilon}XB\ \right\|_2$$

In fact, as usual we may assume that A is a diagonal matrix and we compute

$$\left\|A^{1/1+\varepsilon}XB^{1/1+\varepsilon}\right\|_2^2 = \sum_{i,j}\left|\lambda_i{}^{1/1+\varepsilon}\lambda_j{}^{1/1+\varepsilon}X_{i,j}\right|^2 \leq \sum_{i,j}\left|\left(\frac{\lambda_i+\lambda_j}{1+\varepsilon}\right)X_{i,j}\right|^2 = \left\|\frac{1}{1+\varepsilon}AX + \frac{1}{1+\varepsilon}\right\|_2^2$$

The usual 2×2 matrix argument thus takes care of the rest. On the other hand, for example we have the estimate with the constant $K_{1+\varepsilon}$ (i.e., the one in Theorem 6 or Corollary 7 for the trace-class norm$\|\cdot\|_1$or the ordinary operator norm $\|\cdot\|$. Therefore, the standard interpolation argument gives rise to a better

estimate for the $C_{1+\varepsilon}$-norm $\|\cdot\|_{1+\varepsilon}$.The arguments presented certainly work if operators are matrix. On the other hand, when we deal with Hilbert space operators, we have to be more careful.
We recall the function $f(z) = A^{1+iz}XB^{-iz}$, and for vectors the function $z \to (f(z)\xi, \eta) \in$ C is certainly a bounded continuous function on the strip Ω which is analytic in the interior. Therefore, we have integral expressions such as

$$(A^{1/2}XB^{1/2}\xi, \eta) = \int_{-\infty}^{\infty} (A^{it}(AX + XB)B^{-it}\xi, \eta) f_{1+\varepsilon}(t) \frac{dt}{2\cosh(\pi t)}$$

This means that$Y_n = \int_{-n}^{n} A^{it}(AX + XB)B^{-it}\, dt/2\cosh(\pi t)$converges to$A^{1/2}XB^{1/2}$ in the weak operator topology as $n \to \infty$Since$|||\cdot|||$ is lowersemi-continuous relative to this topology (Proposition 2.2, [9]), we have

$$|||A^{1/2}XB^{1/2}||| \leq \lim_{n\to\infty} \inf |||Y_n|||.$$

Therefore, (to get Theorem (4.2.5), for operators for instance) it is sufficient to show$|||Y_n||| \leq \frac{1}{2}|||AX + XB|||$ for each n. For each fixed n we set $\delta_m = 2n/m$ and consider the operator Riemann sum

$$Z_m = \delta_m \times \sum_{k=0}^{m-1} A^{i(-n+k\delta_m)}(AX + XB)B^{-i(-n+k\delta_m)} \frac{1}{2\cosh(\pi(-n + k\delta_m))}.$$

Everything is continuous so that $(Z_m\xi, \eta)$ converges to $(Y_n\xi, \eta)$ as m→∞This means that Z_m converges to Y_n in the weak operator topology, andthe lower semi-continuity once again implies

$$|||Y_n||| \leq \lim_{m\to\infty} \inf |||Z_m||| \leq \left(\lim_{m\to\infty} \inf \sum_{k=1}^{m-1} \frac{\delta_m}{2\cosh(\pi(-n + k\delta_m))} \right) \times |||AX + XB|||.$$

However, the last coefficient is obviously

$$\int_{-n}^{n} \frac{dt}{2\cosh(\pi t)} \left(\leq \int_{-\infty}^{\infty} \frac{dt}{2\cosh(\pi t)} = \frac{1}{2} \right),$$

and we are done.
Therefore, if$AX + XB$ belongs to the operator ideal determined by $|||\cdot|||$, then so does $A^{1/2}X\,B^{1/2}$ and $|||A^{1/2}X\,B^{1/2}||| \leq \frac{1}{2}|||AX + XB|||$.Other inequalities (except Theorem 5 and Theorem 6 can be justified by the same argument.
Theorem 5 is also valid for operators as long as H, K are compact self-adjoint operators. Recall that we may and do assume H=K by the 2×2- matrix trick, and let s be the support projection of H acting on a Hilbert space $\mathcal{H}$. Let $\{e_i\}_{i=1,2,\ldots}$(resp. $\{f_i\}_{i=1,2,\ldots}$) be an orthonormal basis for s$\mathcal{H}$diagonalizing sH (resp. for the orthogonal complement $(1-s)\mathcal{H}$). Weset

$$(1+\varepsilon)_n = \sum_{i=1}^{n} (e_i \bigotimes e_i^c + f_i \otimes f_i^c) \qquad and \qquad \mathrm{H}_n = (1+\varepsilon)_n \mathrm{H}.$$

We observe that $\sin(H_n)$, $\cos(H_n)$converge to $\sin(H\ \ )$, $\sin(H\ \ )$ respectively innorm and that ($(1+\varepsilon)_n X (1+\varepsilon)_n$converges to$X$in the strong operator topology (due tothe presence of $f_i \otimes f_i^{c\prime}s$ in $(1+\varepsilon)_n$). The operators $\mathrm{H}_n = (1+\varepsilon)_{n_n} \sin Hand(1+\varepsilon)_n X (1+\varepsilon)_n$certainly live on the finite

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dimensional space $(1+\varepsilon)_n$H, but $\cos(H_n)$does not because of $\cos 0 = 1$. However, notice $\cos(H_n) = (1+\varepsilon)_n \cos(H_n) \oplus 1_{\varepsilon\,\mathcal{H}}$and

$$\sin(H_n)\,(1+\varepsilon)_n X(1+\varepsilon)_n \cos(H_n) - \cos(H_n)\,((1+\varepsilon)_n X(1+\varepsilon)_n) \sin(H_n)$$
$$= (\sin(H_n)\,((1+\varepsilon)_n X(1+\varepsilon)_n)(1+\varepsilon)_n) \cos(H_n)$$
$$- (1+\varepsilon)_n) \cos(H_n)\,((1+\varepsilon)_n X(1+\varepsilon)_n) \sin(H_n)) \oplus 0_{\varepsilon\,\mathcal{H}_n}$$

This operator converges to $(\sin H)\,X(\cos H)X(\cos H)\,X(\sin H)$ in the strong operator topology, and note that $(1+\varepsilon)_n)\cos((1+\varepsilon)_n))\,(= (1+\varepsilon)_n)\cos(H_n)(1+\varepsilon)_n))$isnothing but the functional calculus (under$cos(.)$) of the matrix $(1+\varepsilon)_n)$H considered as an operator on $(1+\varepsilon)_n)\mathcal{H}$. Therefore, we have

$$|||(\sin H)X(\cos H) - (\cos H)X(\sin H)|||$$
$$\leq \lim_{n\to\infty} \inf|||\sin(H_n)\,((1+\varepsilon)_n X(1+\varepsilon)_n) \cos(H_n)$$
$$- \cos((1+\varepsilon)_n)H_n)\,((1+\varepsilon)_n X(1+\varepsilon)_n) \sin(H_n)|||$$
$$= \lim_{n\to\infty} \inf|||\sin(H_n)\,((1+\varepsilon)_n X(1+\varepsilon)_n)\,(1+\varepsilon)_n)\cos(H_n)$$
$$- (1+\varepsilon)_n)\cos(H_n)\,((1+\varepsilon)_n X(1+\varepsilon)_n) \sin(H_n)|||$$
$$\leq \lim_{n\to\infty} \inf|||H_n(1+\varepsilon)_n X(1+\varepsilon)_n - (1+\varepsilon)_n X(1+\varepsilon)_n H_n|||$$
$$= \lim_{n\to\infty} \inf|||(1+\varepsilon)_n)(HX - XH)(1+\varepsilon)_n)|||$$

Since$|||(1+\varepsilon)_n(HX-XH)(1+\varepsilon)_n||| \leq \|(1+\varepsilon)_n\|\,|||HX-XH|||\,\|(1+\varepsilon)_n\| \leq |||HX-XH|||$ by theunitary invariance, we obtain Theorem5 for infinite dimensional operators.

Note that Theorem 6 can be dealt in the same way.

The Fourier transforms of the functions $a_{1-\varepsilon}(s), b_{1-\varepsilon}(s)$ are of course well-known. In fact, they can be found in a standard table on Fourier transforms. We also compute the inverse Fourier transform of the function$((1/1+\varepsilon)\exp(s/1+\varepsilon) + (1/1+\varepsilon)\exp(-s/1+\varepsilon))^{-1}$, i.e., the function $f_{1+\varepsilon}(t)$appearing in Theorem6 .

Lemma 8: For $0 < \varepsilon < 1$ we have

$$(\mathcal{g}a_{1-\varepsilon})(s) = \frac{\sinh(\varepsilon s)}{\sinh(s)} \quad and \quad (\mathcal{g}b_{1-\varepsilon})(s) = \frac{\sinh((1-\varepsilon)s)}{\sinh(s)}$$

For a fixed s ∈ R, the function $z \to \exp(isz)$ is a bounded con-tinuous function on the trip Ω which is analytic in the interior. Therefore,the integral formula we have used repeatedly shows

$$exp\,(-(1-\varepsilon)s) = \int_{-\infty}^{\infty} \exp(ist)a_{1-\varepsilon}(t)dt + \int_{-\infty}^{\infty} \exp(ist-s)b_{1-\varepsilon}(t)dt$$

$$= (\mathcal{g}a_{1-\varepsilon})(s) + \exp(-s)\,(\mathcal{g}b_{1-\varepsilon})(s)$$

Similarly, from the function $z \to \exp(-isz)$ we get

$$\exp(\theta s) = \int_{-\infty}^{\infty} \exp(-ist)\,a_{1-\varepsilon}(t)dt + \int_{-\infty}^{\infty} \exp(-ist+s)\,b_{1-\varepsilon}(t)dt$$
$$= (\mathcal{g}a_{1-\varepsilon})(-s) + \exp(s)\,(\mathcal{g}b_{1-\varepsilon})(-s).$$

We then take the conjugate of the second equation. By noting$\exp(\theta s), \exp(s)\epsilon\boldsymbol{R}$and the positive definiteness of $(\mathcal{g}a_{1-\varepsilon})(s), (\mathcal{g}b_{1-\varepsilon})(s)$(hence $\overline{(\mathcal{g}a_{1-\varepsilon})(-s)}$=$(\mathcal{g}a_{1-\varepsilon})(s)$, etc.), we have

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$\exp(s(1-\varepsilon) = (\mathcal{g}a_{1-\varepsilon})(s) + \exp(s)\,(\mathcal{g}b_{1-\varepsilon})(s).$

By solving this equation and the first for $(\mathcal{g}a_{1-\varepsilon})(s)$and $(\mathcal{g}b_{1-\varepsilon})(s)$, we obviously get the desired result. (Note that we cannot solve equations when s=0, but the Fourier transforms$(\mathcal{g}a_{1-\varepsilon})(s)$,$(\mathcal{g}b_{1-\varepsilon})(s)$, are smooth enough.) .

This lemma in particular says

$$\int_{-\infty}^{\infty} a_{1-\varepsilon}(t)dt = (\mathcal{g}a_{1-\varepsilon})(0) = \lim_{s\to 0}\frac{\sinh(\varepsilon s)}{\sinh(s)} = \varepsilon$$

$$\int_{-\infty}^{\infty} b_{1-\varepsilon}(t)dt = (\mathcal{g}b_{1-\varepsilon})(0) = \lim_{s\to 0}\frac{\sinh(\theta s)}{\sinh(s)} = 1-\varepsilon.$$

We next determine the function $f_{1+\varepsilon}(t)$, i.e., the inverse Fourier transform ofthe function $\big((1/1+\varepsilon)\exp(s/1+\varepsilon) + (1/1+\varepsilon)\exp(-s/1+\varepsilon)\big)^{-1}$. Starting from the formula for the Fourier transform of $(2\cos(\pi t))^{-1}\left(= a_{1/2}(t) = b_{1/2}(t)\right)$,wecompute

$$\frac{1}{2\cosh\left(\frac{s}{2}\right)} = \int_{-\infty}^{\infty}\frac{\exp(ist)}{2\cosh(\pi t)}dt = \int_{-\infty}^{\infty}\frac{\exp(ist)\exp(\pi t)}{\exp(2\pi t)+1}dt = \int_{0}^{\infty}\frac{\exp\left(is\times\frac{\log x}{\pi}\right)}{x^2+1}\frac{dx}{\pi},$$

Which means

$$\int_{0}^{\infty}\frac{\exp(is\log x)}{x^2+1}dx = \frac{\pi}{2\cosh\left(\frac{\pi s}{2}\right)}.$$

From this we easily get the formula

$$\int_{0}^{\infty}\frac{\exp(is\log x)}{x^2+a^2}dx = \frac{a^{is-1}\pi}{2\cosh\left(\frac{\pi s}{2}\right)} for\ \ a>0 \qquad (1)$$

by the obvious change of variables.We notice

$$\left(\frac{1}{1+\varepsilon}(\exp\left(\frac{s}{1+\varepsilon}\right) + \exp\left(-\frac{s}{1+\varepsilon}\right)\right)^{-1}) = 0 \qquad (2)$$

With

$$g\,(s) = \left(\tfrac{1}{1+\varepsilon}\exp\left(\tfrac{s}{1+\varepsilon}\right) + \tfrac{1}{1+\varepsilon}\exp\left(-\tfrac{s}{1+\varepsilon}\right)\right)^{-1}$$

and we begin by computing the inverse Fourier transform of g(s). Wecompute

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp(-ist)\,g(s)ds = \frac{1+\varepsilon}{2\pi}\int_{-\infty}^{\infty}\frac{\exp(-ist)\exp\left(\frac{s}{2}\right)}{\exp(s)+1}ds$$

$$= \frac{1+\varepsilon}{2\pi}\int_{0}^{\infty}\frac{\exp(-it\times 2\log x)}{x^2+1}2dx$$

$$= \frac{1+\varepsilon}{\pi}\int_{0}^{\infty}\frac{\exp(-2it\log x)}{x^2+1}dx$$

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Hence, thanks to the formula (1) (with a 1 and s=−2t) we get

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp(-it)\,g(1)ds = \frac{1+\varepsilon}{\pi}\times\frac{1}{2\cosh(\pi t)} = \frac{1+\varepsilon}{2}\times\frac{1}{\cosh(\pi t)}$$

We actually need

Lemma 9:. As long as $|\Im z|<\frac{1}{2}$, we have

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp(-isz)\,g(s)ds = \frac{1+\varepsilon}{2}\times\frac{1}{\cosh(\pi z)}$$

Proof: Recalling the definition of the function g(s) (≥0), we observe

$$\int_{-\infty}^{\infty}\exp(\delta s)\,g(s)ds < \infty \quad as\ long\ as \quad |\delta| < \frac{1}{2}$$

From this it is straightforward to see that the function $z \to \int_{-\infty}^{\infty}\exp(-isz)$g (s) $ds$is analytic on the open strip $-\frac{1}{2} < Iz < \frac{1}{2}$thanks toLebesgue's dominated convergence theorem. On the other hand, the zerosof the entire function $\cosh(\pi z)$are$\{(\frac{1}{2}+n)i\}_{n\in Z}$. In particular, the function $1+\varepsilon/2\times\frac{1}{\cosh(\pi z)}$ is analytic on the same open strip. The computation right before the lemma guarantees that the two functions arethe same on the real line so that the result follows from the identity theorem for analytic functions.

Lemma10:. We have

$$f_{\varepsilon+1}(t) = \frac{(\varepsilon+1)^{2/\varepsilon+1}}{2(\cosh(\pi t))}$$

Proof: We compute

$$\begin{aligned} f_{\varepsilon+1}(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\exp(-ist)\left(\frac{1}{\varepsilon+1}\exp\left(\frac{s}{\varepsilon+1}\right)+\frac{1}{\varepsilon+1}\exp\left(-\frac{s}{\varepsilon+1}\right)\right)^{-1}ds \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\exp(-ist)\,g(s\,)ds \qquad (by(9)) \\ &= \frac{\varepsilon+1}{2}\times\frac{1}{\cosh\pi t} \\ &= \frac{\varepsilon+1}{2}\times\frac{1}{\cosh \pi t}. \end{aligned}$$

The last coefficient in the preceding expression is $\frac{1}{2}(1+\varepsilon)^{\frac{2}{1+\varepsilon}}$ while the denominator is equal to$cos(\pi t)$.

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