

The Operators of Composition-Differentiation on the Hardy Space

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Abstract.

Suppose φ_j be a nonconstant analytic self-map of the open unit disk in $\mathbb{C}$, with $\sum \|\varphi_j\|_\infty < 1$. Regard the operators D_{φ_j} , acting on the Hardy space H^2 , given by differentiation followed by composition with φ_j . Mahsa Fatehi and Christopher N. B. Hammond. Obtain results relating to the adjoint, norm, and spectrum of such an operator.

Key words: Operator, Differentiation, Hardy Space, analytic functions, composition–differentiation, open unit disk

1. Preliminaries

Suppose $\mathbb{D}$ denote the open unit disk in the complex plane. The Hardy space H^2 is the Hilbert space consisting of every analytic function $\sum f_j(z) = \sum_{n=0}^{\infty} a_n^j z^n$ on $\mathbb{D}$ such that

$$\sum \|f_j\|^2 = \sum_{n=0}^{\infty} \sum |a_n^j|^2 < \infty.$$

For $\sum f_j(z) = \sum_{n=0}^{\infty} a_n^j z^n$ and $\sum g^j(z) = \sum_{n=0}^{\infty} b_n^j z^n$ in H^2 , their inner product is defined

$$\sum \langle f_j, g^j \rangle = \sum_{n=0}^{\infty} a_n^j \overline{b_n^j}.$$

Write H^∞ to denote the space of every bounded analytic function on $\mathbb{D}$, with $\sum \|f_j\|_\infty = \sup \{ \sum |f_j(z)| : z \in \mathbb{D} \}$.

For an analytic map $\varphi_j: \mathbb{D} \rightarrow \mathbb{D}$, the composition operators C_{φ_j} are defined

$$C_{\varphi_j}(f_j) = f_j \circ \varphi_j.$$

Every composition operators are bounded on H^2 , with $\sum \sqrt{\frac{1}{1-|\varphi_j(0)|^2}} \leq \sum \|C_{\varphi_j}\| \leq \sum \sqrt{\frac{1+|\varphi_j(0)|}{1-|\varphi_j(0)|}}$.

(See, for example, [2, Corollary 3.7].) For a function ψ_j in H^∞ , the Toeplitz operators $T_{\psi_j}^j$ are defined $T_{\psi_j}^j(f_j) = \psi_j \cdot f_j$.

Every such operators are bounded on H^2 , with $\|T_{\psi_j}^j\| = \|\psi_j\|_\infty$ (see [6, Theorem 5]).

Abusing notation somewhat, we will write T_z^j to indicate the operators $T_{\psi_j}^j$ for $\psi_j(z) = z$. Observe that T_z^j is an isometry on H^2 .

In the context of analytic functions on $\mathbb{D}$, it also appears reasonable to peeking operators defined in terms of differentiation. It is easy to see that the differentiation operators $D(f_j) = \dot{f}_j$ are unbounded on the Hardy space: $\|D(z^n)\|/\|z^n\| = n$ for any natural number n . Nevertheless, for many analytic maps $\varphi_j: \mathbb{D} \rightarrow \mathbb{D}$, the operators

$$f_j(z) \mapsto \dot{f}_j(\varphi_j(z)) \quad (1.1)$$

Received: 12/08/2024

Accepted: 30/09/2024

is bounded on H^2 . Some authors, following the example of [4] and [5], have used the notation $C_{\varphi_j}D$ to denote such operators. Because of the unboundedness of D , it makes sense to write (1.1) as a single operator, specifically when that operator is bounded on H^2 . Write D_{φ_j} to denote the operators on H^2 given by $D_{\varphi_j}(f_j) = \hat{f}_j \circ \varphi_j$.

In this paper we will refer to such an operator as a composition–differentiation operator. The Closed Graph Theorem exhibits that D_{φ_j} are bounded on H^2 whenever D_{φ_j} takes H^2 into itself.

Ohno [5] established a basic set of results relating to when the operators we are calling D_{φ_j} are bounded or compact on H^2 . We will only be considering φ_j with $\sum \|\varphi_j\|_{\infty} < 1$, in which case D_{φ_j} is guaranteed to be Hilbert–Schmidt on H^2 , and therefore both bounded and compact (see [5, Theorem 3.3]). There are cases of bounded or compact D_{φ_j} with $\sum \|\varphi_j\|_{\infty} = 1$, but they are beyond the scope of the current investigation.

The purpose of this note is to explore the operators D_{φ_j} in more detail. In specific, we find a representation for the adjoint $D_{\varphi_j}^*$ when φ_j is linear fractional (Theorem 1). In the particular case where $\rho_j(z) = r_j z$ for $0 < |r_j| < 1$, we compute the norm $\|D_{\rho_j}\|$ explicitly (Theorem 2) and exhibit that D_{ρ_j} is quasiniipotent (Proposition 3). Applying established results relating to composition operators, also obtain estimates for the norm of D_{φ_j} whenever $\sum \|\varphi_j\|_{\infty} < 1$ (Proposition 4).

For any point w_j in $\mathbb{D}$, define $K_{w_j}(z) = \frac{1}{1-\overline{w_j}z}$.

It is well known that K_{w_j} acts as the reproducing kernel function for point-evaluation:

$$\sum \langle f_j, K_{w_j} \rangle = \sum f_j(w_j)$$

for any f_j in H^2 . Therefore $\sum \|K_{w_j}\| = \sum \sqrt{K_{w_j}(w_j)} = \sum \sqrt{\frac{1}{1-|w_j|^2}}$.

In identical manner, define $K_{w_j}^{(1)}(z) = \frac{z}{(1-\overline{w_j}z)^2}$.

Notice that $K_{w_j}^{(1)}$ acts as the reproducing kernel for point-evaluation of the first derivative:

$$\sum \langle f_j, K_{w_j}^{(1)} \rangle = \sum f_j'(w_j)$$

for any f_j in H^2 (see [2, Theorem 2.16]). In specific, $\sum \|K_{w_j}^{(1)}\| = \sum \sqrt{(K_{w_j}^{(1)})'(w_j)} =$

$$\sum \sqrt{\frac{1+|w_j|^2}{(1-|w_j|^2)^3}}$$

for any w_j in $\mathbb{D}$.

It is well known that $C_{\varphi_j}^*(K_{w_j}) = K_{\varphi_j(w_j)}$ and $(T_{\psi_j}^j)^*(K_{w_j}) = \overline{\psi_j(w_j)} K_{w_j}$ for any w_j in $\mathbb{D}$. Identical property holds for any bounded operators D_{φ_j} on H^2 . Observe that

$$\sum \langle f_j, D_{\varphi_j}^*(K_{w_j}) \rangle = \sum \langle D_{\varphi_j}(f_j), K_{w_j} \rangle = \sum f_j'(\varphi_j(w_j)) = \sum \langle f_j, K_{\varphi_j(w_j)}^{(1)} \rangle$$

Received: 12/08/2024

Accepted: 30/09/2024

for every f_j in H^2 , so $D_{\varphi_j}^*(K_{w_j}) = K_{\varphi_j(w_j)}^{(1)}$ for any w_j in $\mathbb{D}$. In specific,

$$\sum \frac{\|D_{\varphi_j}^*(K_{w_j})\|}{\|K_{w_j}\|} = \sqrt{\sum \frac{(1-|w_j|^2)(1+|\varphi_j(w_j)|^2)}{(1-|\varphi_j(w_j)|^2)^3}}. \quad (1.2)$$

Therefore, if D_{φ_j} is bounded on H^2 , we see that

$$\sum \|D_{\varphi_j}\| \geq \sup_{w_j \in \mathbb{D}} \sqrt{\sum \frac{(1-|w_j|^2)(1+|\varphi_j(w_j)|^2)}{(1-|\varphi_j(w_j)|^2)^3}} \geq \sup_{w_j \in \mathbb{D}} \sqrt{\sum \frac{1-|w_j|^2}{(1-|\varphi_j(w_j)|^2)^3}}$$

$$\text{And } \sum \|D_{\varphi_j}\|_e \geq \limsup_{|w_j| \rightarrow 1} \sqrt{\sum \frac{(1-|w_j|^2)(1+|\varphi_j(w_j)|^2)}{(1-|\varphi_j(w_j)|^2)^3}} \geq$$

$$\limsup_{|w_j| \rightarrow 1} \sqrt{\sum \frac{1-|w_j|^2}{(1-|\varphi_j(w_j)|^2)^3}}. \quad (1.3)$$

These findings should be compared to [5, Corollary 3.2], which outlines the necessary and sufficient requirements for D_{φ_j} to be bounded or compact when φ_j is univalent. In specific, (1.3) exhibits that D_{φ_j} cannot be bounded if φ_j has finite angular derivative at any point on $\partial\mathbb{D}$ (see [2, Theorem 2.44]). Taking $w_j = 0$ in (1.2), we see

$$\text{that } \sum \|D_{\varphi_j}\| \geq \sqrt{\sum \frac{1+|\varphi_j(0)|^2}{(1-|\varphi_j(0)|^2)^3}},$$

which exhibits that $\sum \|D_{\varphi_j}\| \geq 1$ for each φ_j and that $\sum \|D_{\varphi_j}\| > 1$ whenever $\varphi_j(0) \neq 0$.

For the duration of this note, we will assume that φ_j is a nonconstant analytic map with $\sum \|\varphi_j\|_\infty < 1$. As mentioned earlier, this assumption guarantees that D_{φ_j} is compact on H^2 .

2. Adjoints, Norms, and Spectra

We obtain information about the adjoint, norm, and spectrum of D_{φ_j} in certain specific

cases. If $\varphi_j(z) = \frac{a^j z + b^j}{c^j z + d^j}$

is a non constant linear fractional self-map of $\mathbb{D}$, hence the map $\sigma_j(z) = \frac{\overline{a^j} z - \overline{c^j}}{-\overline{b^j} z + \overline{d^j}}$ also takes $\mathbb{D}$ into itself (see [1, Lemma 1]). It is not difficult to exhibit that $\sum \|\sigma_j\|_\infty < 1$ whenever $\sum \|\varphi_j\|_\infty < 1$. The link between these two maps has long been regarded in reference to the adjoints of composition operators.

Theorem 1. For a pair of linear fractional maps φ_j and σ_j , as described above,

$$D_{\varphi_j}^* \left(T_{K_{\sigma_j(0)}}^j \right)^* = T_{K_{\varphi_j(0)}}^j D_{\sigma_j}$$

Proof. Observe that $K_{\varphi_j(0)}^{(1)}(z) = \frac{z}{(1-(\overline{b^j}/d^j)z)^2} = \frac{(\overline{d^j})^2 z}{(-\overline{b^j} z + \overline{d^j})^2}$

Received: 12/08/2024

Accepted: 30/09/2024

And $K_{\sigma_j(0)}^{(1)}(z) = \frac{z}{(1+(c^j/d^j)z)^2} = \frac{(d^j)^2 z}{(c^j z + d^j)^2}$.

$$\begin{aligned} \text{Furthermore, see that; } T_{K_{\sigma_j(0)}^{(1)}}^j D_{\sigma_j}(K_{w_j})(z) &= T_{K_{\sigma_j(0)}^{(1)}}^j \frac{\overline{w_j}}{\left(1 - \overline{w_j} \left(\frac{\overline{a^j z - c^j}}{-b^j z + d^j}\right)\right)^2} \\ &= \frac{(\overline{d^j})^2 z}{(-\overline{b^j} z + \overline{d^j})^2} \cdot \frac{\overline{w_j}}{\left(1 - \overline{w_j} \left(\frac{\overline{a^j z - c^j}}{-b^j z + d^j}\right)\right)^2} \\ &= \frac{(\overline{d^j})^2 \overline{w_j} z}{(-\overline{b^j} z + \overline{d^j} - \overline{a^j} \overline{w_j} z + c^j \overline{w_j})^2} \\ &= \frac{(\overline{d^j})^2 \overline{w_j}}{(c^j \overline{w_j} + \overline{d^j})^2} \cdot \frac{z}{\left(1 - \left(\frac{\overline{a^j} \overline{w_j} + b^j}{c^j \overline{w_j} + d^j}\right) z\right)^2} \end{aligned}$$

for any w_j in $\mathbb{D}$. Second hand,

$$D_{\varphi_j}^* \left(T_{K_{\sigma_j(0)}^{(1)}}^j \right)^* (K_{w_j}) = \frac{(\overline{d^j})^2 \overline{w_j}}{(c^j \overline{w_j} + \overline{d^j})^2} D_{\varphi_j}^* (K_{w_j}) = \frac{(\overline{d^j})^2 \overline{w_j}}{(c^j \overline{w_j} + \overline{d^j})^2} K_{\varphi_j(w_j)}^{(1)}.$$

Therefore, $D_{\varphi_j}^* \left(T_{K_{\sigma_j(0)}^{(1)}}^j \right)^*$ and $T_{K_{\sigma_j(0)}^{(1)}}^j D_{\sigma_j}$ agree on the span of the reproducing kernelfunctions, which constitutes a dense subset of H^2 . Therefore, the two operators are identical on H^2 .

This result bears a close resemblance to Cowen's adjoint formula for composition operators (see [1, Theorem 2]), which can be rewritten $C_{\varphi_j}^* \left(T_{K_{\sigma_j(0)}^{(1)}}^j \right)^* = T_{K_{\sigma_j(0)}^{(1)}}^j C_{\sigma_j}$.

Let us focus on the specific situation where $\rho_j(z) = r_j z$ for a real number r_j . In

this case, Theorem 1 reduces to $D_{\rho_j}^* (T_z^j)^* = T_z^j D_{\rho_j}$. Therefore

$$D_{\rho_j}^* = D_{\rho_j}^* (T_z^j)^* T_z^j = T_z^j D_{\rho_j} T_z^j,$$

and therefore $D_{\rho_j} D_{\rho_j}^* = D_{\rho_j} T_z^j D_{\rho_j} T_z^j = (D_{\rho_j} T_z^j)^2$.

Note that; $(D_{\rho_j} T_z^j)^* = (T_z^j)^* D_{\rho_j}^* = (T_z^j)^* T_z^j D_{\rho_j} T_z^j = D_{\rho_j} T_z^j$,

so $D_{\rho_j} T_z^j$ is self-adjoint. If $0 < r_j < 1$, we will see that $D_{\rho_j} T_z^j = |D_{\rho_j}^*|$. In specific, if we can determine the spectrum of $D_{\rho_j} T_z^j$, we will be able to compute $\|D_{\rho_j}\|$.

Theorem 2. If $\rho_j(z) = r_j z$ for some real number $0 < r_j < 1$, hence

$$\Sigma \|D_{\rho_j}\| = \Sigma \left\lfloor \frac{1}{1-r_j} \right\rfloor r_j^{[1/(1-r_j)]-1}, \quad 2.1$$

where $\lfloor \cdot \rfloor$ denotes the greatest integer function.

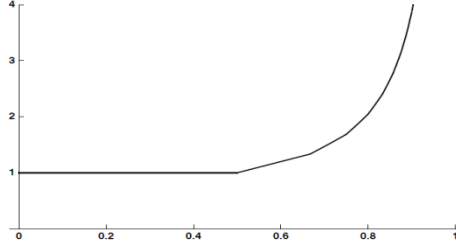


Figure 1. The quantity $\|D_{\rho_j}\|$ for different values of r_j .

Proof. Note that $D_{\rho_j} T_z^j(z^{n-1}) = D_{\rho_j}(z^n) = n(r_j z)^{n-1} = nr_j^{n-1} z^{n-1}$

for any natural number n . Therefore, the spectrum of $D_{\rho_j} T_z^j$ includes $\{nr_j^{n-1} : n \in \mathbb{N}\}$.

Second hand, suppose λ_j be an arbitrary eigenvalue for $D_{\rho_j} T_z^j$ with corresponding eigenvector f_j . Observe that; $\lambda_j f_j(z) = f_j(r_j z) + r_j z f_j'(r_j z)$.

If $f_j(0) \neq 0$, hence $\lambda_j = 1$. If $f_j(0) = 0$, differentiate both sides of the equation above to obtain $\lambda_j f_j'(z) = 2r_j f_j'(r_j z) + r_j^2 z f_j''(r_j z)$.

If $f_j'(0) \neq 0$, hence $\lambda_j = 2r_j$. Differentiating a total of $n - 1$ times, we obtain

$$\lambda_j f_j^{(n-1)}(z) = nr_j^{n-1} f_j^{(n-1)}(r_j z) + r_j^n z f_j^{(n)}(r_j z).$$

If $f_j^{(n-1)}(0) \neq 0$, Therefore $\lambda_j = nr_j^{n-1}$. Therefore, any eigenvalue must have the form nr_j^{n-1} for some natural number n . Since $D_{\rho_j} T_z^j$ is compact, its spectrum is exactly $\{0\} \cup \{nr_j^{n-1} : n \in \mathbb{N}\}$.

Since all the eigen values are positive, see that $\sum D_{\rho_j} T_z^j = \sum |D_{\rho_j}^*|$. Therefore,

$$\sum \|D_{\rho_j}\| = \max \left\{ \sum nr_j^{n-1} : n \in \mathbb{N} \right\}.$$

To obtain (2.1), observe that the function $h_j(x) = xr_j^{n-1}$ has precisely one local extremum on $[0, \infty)$, which are also its absolute maximum. Therefore we simply need to find the greatest natural number n so that $(n - 1)r_j^{n-2} \leq nr_j^{n-1}$

or equivalently $1 - \frac{1}{n} \leq r_j$

or $n \leq \frac{1}{1-r_j}$.

Thus the quantity nr_j^{n-1} is maximized when $n = \lfloor 1/(1 - r_j) \rfloor$.

There are several interesting consequences of Theorem 2. First of each, $\sum \|D_{\rho_j}\| = 1$ for $0 < r_j \leq 1/2$ and $\sum \|D_{\rho_j}\| > 1$ for $1/2 < r_j < 1$. Secondly, $\sum \|D_{\rho_j}\|$ tends to ∞ as r_j goes to 1. Since composition with a rotation is an isometry, (2.1) holds with r_j replaced by $|r_j|$ for any complex number r_j with $0 < \sum |r_j| < 1$. Likewise, the same formula holds for $\sum \|D_{\varphi_j}\|$ where $\varphi_j(z) = r_j z^k$ for any k in $\mathbb{N}$.

We deduce this section with an observation relating to the spectrum of D_{ρ_j} , which indeed pertains to a slightly larger class of operators.

Received: 12/08/2024

Accepted: 30/09/2024

Proposition 3. If $\varphi_j(z) = a^j z + b^j$, where $0 < \sum |a^j| < 1 - \sum |b^j|$, the operators D_{φ_j} are quasinilpotent; that is, its spectrum is $\{0\}$.

Proof. Assume that λ_j are a nonzero element of the spectrum of D_{φ_j} , which would necessarily be an eigenvalue. Hence there is a nonzero function f_j in H^2 such that;

$$\lambda_j f_j(z) = f_j'(a^j z + b^j).$$

It is clear from this formula that f_j cannot be a polynomial. Differentiating n times, see that

$$\lambda_j f_j^{(n)}(z) = (a^j)^n f_j^{(n+1)}(a^j z + b^j),$$

so that $(\lambda_j / (a^j)^n) f_j^{(n)}(z) = (f_j^{(n)})'(a^j z + b^j)$.

Since $f_j^{(n)}$ are not identically 0, we deduce that $\lambda_j / (a^j)^n$ must be an eigenvalue for D_{φ_j} for every n , which is impossible. Therefore $\lambda_j = 0$ are the only elements in the spectrum of D_{φ_j} .

Remark. Make a stronger statement than Proposition 3. When $\varphi_j(z) = a^j z + b^j$, note that $D_{\varphi_j}^n$ are never equal to the trivial operators on H^2 . Furthermore, for any natural number M , the closed linear span of the set $\{z^n\}_{n=0}^M$ is invariant under the operators D_{φ_j} . Therefore [3, Corollary 1] exhibits that D_{φ_j} are a compact universal quasinilpotent operators; that is, the norm closure of $\{W D_{\varphi_j} W^{-1} : W \text{ is an invertible operator on } H^2\}$

contains every compact quasinilpotent operator.

3. Connections with composition operators

We can view the operators D_{φ_j} as the product $C_{\varphi_j} D_{\rho_j}$, although the benefit of this representation is limited by the fact that D is unbounded on H^2 . The bounded operators D_{ρ_j} , where $\rho_j(z) = r_j z$ for $0 < r_j < 1$, can act as a surrogate for D in the situation where $\sum \|\varphi_j\|_{\infty} < 1$. Recall that D_{ρ_j} are compact and quasinilpotent, and that we know both its adjoint and its norm.

Take $\sum \|\varphi_j\|_{\infty} \leq \sum r_j < 1$ and define $(\varphi_j)_{r_j} = (1/r_j)\varphi_j$. Observe that $D_{\varphi_j} = C_{(\varphi_j)_{r_j}} D_{\rho_j}$.

(3.1)
Since $\sum \|D_{\varphi_j}\| = \sum \|C_{(\varphi_j)_{r_j}}\| \|D_{\rho_j}\|$, obtain the following estimate for $\|D_{\varphi_j}\|$.

Proposition 4. If φ_j are a nonconstant analytic self-map of $\mathbb{D}$ with $\sum \|\varphi_j\|_{\infty} < 1$, then

$$\sum \sqrt{\frac{1 + |\varphi_j(0)|^2}{(1 - |\varphi_j(0)|^2)^3}} \leq \sum \|D_{\varphi_j}\| \leq \sum \sqrt{\frac{r_j + |\varphi_j(0)|}{r_j - |\varphi_j(0)|} \left| \frac{1}{1 - r_j} \right| r_j^{[1/1-r_j]-1}}$$

whenever $\sum \|\varphi_j\|_{\infty} \leq \sum r_j < 1$.

If $\sum \|\varphi_j\|_\infty \leq 1/2$ we may take $r_j = 1/2$ to see that $\sum \sqrt{\frac{1+|\varphi_j(0)|^2}{(1-|\varphi_j(0)|^2)^3}} \leq \sum \|D_{\varphi_j}\| \leq \sum \sqrt{\frac{1+2|\varphi_j(0)|}{1-2|\varphi_j(0)|}}$.

In specific, $\sum \|D_{\varphi_j}\| = 1$ whenever both $\sum \|\varphi_j\|_\infty \leq 1/2$ and $\varphi_j(0) = 0$.

Example 5. Suppose $\varphi_j(z) = a^j z + b^j$, where $a^j \neq 0$ and $\sum |a^j| + \sum |b^j| \leq 1/2$, so that

$$\sum \sqrt{\frac{1+|b^j|^2}{(1-|b^j|^2)^3}} \leq \sum \|D_{\varphi_j}\| \leq \sum \sqrt{\frac{1+2|b^j|}{1-2|b^j|}}.$$

We can indeed improve on Proposition 4 because we know the norm of $C_{(\varphi_j)_{r_j}}$ precisely. For $r_j = 1/2$, it follows from [1, Theorem 3] that

$$\sum \|C_{(\varphi_j)_{r_j}}\| = \sum \sqrt{\frac{2}{1+4|a^j|^2-4|b^j|^2+\sqrt{(1-4|a^j|^2+4|b^j|^2)^2-16|b^j|^2}}}.$$

This value serves as a more precise upper bound for $\|D_{\varphi_j}\|$.

Working with D_{ρ_j} can also be useful for describing the adjoint of operators D_{φ_j} .

Collecting (3.1) with the identity $D_{\rho_j}^* = T_z^j D_{\rho_j} T_z^j$, we see that $D_{\varphi_j}^* = T_z^j D_{\rho_j} T_z^j C_{(\varphi_j)_{r_j}}^*$

whenever $\sum \|\varphi_j\|_\infty \leq \sum r_j < 1$. This notation allows us to determine $D_{\varphi_j}^*$ whenever we know $C_{(\varphi_j)_{r_j}}^*$.

Example 6. Suppose $\varphi_j(z) = r_j z^2$ for $0 < r_j < 1$ and peeking $(\varphi_j)_{r_j}(z) = z^2$. It is well

known (see [2, Exercise 9.1.1]) that $C_{(\varphi_j)_{r_j}}^*(f_j) = \frac{f_j(\sqrt{z}) + f_j(-\sqrt{z})}{2}$

for any f_j in H^2 ; that is, $C_{(\varphi_j)_{r_j}}^*(z^{2n}) = z^n$ and $C_{(\varphi_j)_{r_j}}^*(z^{2n+1}) = 0$ for any

nonnegative integer n . Consequently, $D_{\varphi_j}^*(z^{2n}) = (n+1)r_j^n z^{n+1}$

and $D_{\varphi_j}^*(z^{2n+1}) = 0$. For example, note that

$$\begin{aligned} \sum D_{\varphi_j}^*(\log(1-z)) &= \sum D_{\varphi_j}^*\left(-\sum_{n=1}^{\infty} \frac{z^n}{n}\right) = -\sum_{n=1}^{\infty} \sum_{n=1}^{\infty} \frac{(n+1)r_j^n z^{n+1}}{2n} \\ &= -\frac{z}{2} \sum_{n=1}^{\infty} \sum_{n=1}^{\infty} (r_j z)^n - \frac{z}{2} \sum_{n=1}^{\infty} \sum_{n=1}^{\infty} \frac{(r_j z)^n}{n} \\ &= -\sum \frac{r_j z^2}{2(1-r_j z)} + \sum \frac{z \log(1-r_j z)}{2}. \end{aligned}$$

Observing that every D_{φ_j} from Example 6 has at least one nonzero eigenvalue, since

$$D_{\varphi_j}(z^2) = 2(r_j z^2) = (2r_j)z^2.$$

Results:

1. Ohno [5] established a basic set of results relating to when the operators we are calling D_{φ_j} are bounded or compact on H^2 .



Received: 12/08/2024

Accepted: 30/09/2024

2. Applying established results relating to composition operators, also obtain estimates for the norm of D_{φ_j} whenever $\sum \|\varphi_j\|_{\infty} < 1$ (Proposition 4).

3. This result bears a close resemblance to Cowen's adjoint formula for composition operators (see [1, Theorem 2]), which can be rewritten $C_{\varphi_j}^* \left(T_{K_{\sigma_j(0)}}^j \right)^* = T_{K_{\varphi_j(0)}}^j C_{\sigma_j}$.

Conclusion

In order to determine if the operators we are calling D_{φ_j} are bounded or compact on H^2 , conclusion is quite similar to Cowen's adjoint formula for composition operators, which can be expressed as $C_{\varphi_j}^* \left(T_{K_{\sigma_j(0)}}^j \right)^* = T_{K_{\varphi_j(0)}}^j C_{\sigma_j}$.

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