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# Weighted Composition Operators $(1+\epsilon)\varphi_r$ On Bergman Type Spaces $A^{1+\epsilon}_{z+\epsilon}$ Whit Double Weighted $(z+\epsilon)$

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**Abstract**

The study investigates the boundedness , compactness, essential norm and the Schatten class of weighted composition operators $(1+\epsilon)C_{\varphi_r}$ on Bergman types spaces $A^{1+\epsilon}_{z+\epsilon}$ with double weight $(z+\epsilon)$. Let $X=(1+\epsilon)\in H(\mathbb{D}):(1+\epsilon)C_{\varphi_r}:A^{1+\epsilon}_{z+\epsilon}\to A^{1+\epsilon}_{z+\epsilon}$ is bounded. For some regular weights $(z+\epsilon)$, the researcher obtains that $X=H^\infty$ if and only if $\varphi_r$ is a finite Blaschke product.

**Keywords**: Weighted composition operator, weighted Bergman space, double weight.

**Introduction**

Let $\mathbb{D}$ be the open unit disk in the complex plane, and $H(\mathbb{D})$ the class of all functions analytic on $\mathbb{D}$. Let $\varphi_r$ be an analytic self-map of $\mathbb{D}$ and $(1+\epsilon)\in H(\mathbb{D})$. Juntao Du, Songxino Li and Yecheng Shi [21]. The researcher intends to make few specific changes. The weighted composition operator, denoted by $(1+\epsilon)C_{\varphi_r}$, is defined on $H(\mathbb{D})$ by

$$((1+\epsilon)C_{\varphi_r}f^2)(z)=(1+\epsilon)(z)f^2\big(\varphi_r(z)\big), f^2\in H(\mathbb{D}).$$

For $-1<\epsilon<\infty, H^{1+\epsilon}$ denotes the Hardy space, which consisting of all functions $f^2\in H(\mathbb{D})$ satisfied

$$\|f^2\|^{1+\epsilon}_{H^{1+\epsilon}}=\sup_{0<\epsilon<\infty}\frac{1}{2\pi}\int_0^{2\pi}\left|f^2(1-\epsilon)e^{i\theta}\right|^{1+\epsilon}\mathrm{d}(\theta)<\infty.$$

As usual, $H^\infty$ is the set of bounded analytic functions in $\mathbb{D}$.

The researcher says that $\mu$ is a weight, when $\mu$ radial and positive on $\mathbb{D}$. Suppose that $(z+\epsilon)$ is an integrable weight on (0,1). Let $\widehat{z+\epsilon}(1-\epsilon)\int_{1-\epsilon}^1(z+\epsilon)(1+\epsilon)d(1+\epsilon)$ for $0<\epsilon<\infty$. The researcher says that $z+\epsilon$ is regular, denoted by $(z+\epsilon)\in\mathcal{R}$, if there is a constant $C>0$ depending on $z+\epsilon$, sush that

$$\frac{1}{C}<\frac{\widehat{z+\epsilon}}{\epsilon(z+\epsilon)(1-\epsilon)}<C, \text{when } 0<\epsilon<\infty.$$

The researcher gives that $z+\epsilon$ is rapidly increasing, denoted by$(z+\epsilon)\in I$, if

$\lim_{\epsilon\to 0}\frac{\widehat{z+\epsilon}(1-\epsilon)}{\epsilon(z+\epsilon)(1-\epsilon)}=\infty.$

**Let**

$$\nu_{\alpha,\beta}(1-\epsilon)=\epsilon^\alpha\left(\log\frac{e}{\epsilon}\right)^\beta.$$

After a calculation, The researcher gives the following typical examples of regular and rapidly increasing weights, see [11], for example.

(i) When $\alpha>-1$ and $\beta\in\mathbb{R}, \nu_{\alpha,\beta}\in\mathcal{R}$;

(ii) When $\alpha=-1$ and $\beta<-1$, $\nu_{\alpha,\beta}\in I$ and $\left|\sin\left(\log\frac{1}{\epsilon}\right)\right|\nu_{\alpha,\beta}(2+\epsilon)\in I$.

In [10], Peláez introduced the set of double weights, denoted by $\widehat{\mathcal{D}}$, which includes $I\cup\mathcal{R}$. The researcher says that $(z+\epsilon)\in\widehat{\mathcal{D}}$ if there is a constant $C>0$ such that $\widehat{z+\epsilon}(1-\epsilon)<C(\widehat{z+\epsilon})\left(\frac{2-\epsilon}{2}\right)$, when $0<\epsilon<\infty$. The researcher shoulds remark that the most part of the results see[21], which presented in the context

of regular and rapidly increasing weights, continue to hold for the wider class $\widehat{\mathcal{D}}$. More details about $I, \mathcal{R}$ and $\widehat{\mathcal{D}}$ can be seen in[10,11,13].

For $-1 < \epsilon < \infty$ and $(z+\epsilon)\ \widehat{\mathcal{D}}$, the weighted Bergman space $A^{1+\epsilon}_{Z+\epsilon}$ is the space of $f^2 \in H(\mathbb{D}$ for which

$$\|f^2\|^{1+\epsilon}_{A^{1+\epsilon}_{Z+\epsilon}} = \int |f^2(z)|^{1+\epsilon}\,(z+\epsilon)(z)dA(z) < \infty,$$

Where $dA(z) = \frac{1}{\pi}dxdy$ is the normalized Lebesgue area measure on $\mathbb{D}$. When $(z+\epsilon)(1+\epsilon) = (-\epsilon)^{\alpha}(\alpha > -1)$, the space $A^{1+\epsilon}_{\alpha}$ becomes the classical weighted Bergman space $A^{1+\epsilon}_{\alpha}$. the researcher refers [3,7,20] and references therein. In respects, the Hardy $H^{1+\epsilon}$ is the limit of $A^{1+\epsilon}_{\alpha}$ as $\alpha \to -1$. But it is a rough estimate since most of the finer function- theoretic properties of the classical weighted Bergman space $A^{1+\epsilon}_{\alpha}$ can not carry over to the Hardy space $H^{1+\epsilon}$. the researcher knows, $A^{1+\epsilon}_{z+\epsilon}$ induced by regular weighted have similar properties with $A^{1+\epsilon}_{\alpha}$. But many results see [21] show that spaces $A^{1+\epsilon}_{z+\epsilon}$ induced by rapidly increasing weighted, lie "closer" to $H^{1+\epsilon}$ than any $A^{1+\epsilon}_{\alpha}$.

In [4], Čuckovič and Zhao characterized the boundedness and compactness of weighed composition operators mapping on Bergman space $A^{1+\epsilon}_{\alpha}$ by using Berezin transform. The researcher investigates weighted composition operators between different Bergman spaces and Hardy spaces in [5]. In [13], peláez and Rättyä characterized the Schatten class of Toeplitz operator is a positive Borel measure on $\mathbb{D}$ and the reproducing kernel of the Bergman space $A^2_{z+\epsilon}$ when $(z+\epsilon) \in \widehat{\mathcal{D}}$. In [18], Zhao and Hou proved that, for $A^{1+\epsilon}_{\alpha}$ the finite Blaschke product is the only composition symbol that the induced weighted composition operator is bounded if and only if the weighted sympol defines a bounded multiplication operator. The similar result for Hardy space $H^{1+\epsilon}$ can be seen in [21]. Motivated by [4,5,13], under the assumption that $(z+\epsilon) \in \widehat{\mathcal{D}}$ and $\mu$ is a positive Borel measure, the researcher investigates the boundedness , compactness and essential and norm of $(1+\epsilon)C_{\varphi_r}: A^{1+\epsilon}_{z+\epsilon} \to A^{1+2\epsilon}_{\mu}$ and the Schatten class of $(1+\epsilon)C_{\varphi_r}: A^2_{z+\epsilon} \to A^2_{z+\epsilon.}$ Motivated by [18], the researcher gets that, for some $(z+\epsilon)_* \in \mathcal{R}$, $X = H^{\infty}$ if and only if $\varphi_r$ is a finite Blaschke prouduct. Here

$$X = \{1+\epsilon: (1+\epsilon) \in H(\mathbb{D}) \text{and} (1+\epsilon)C_{\varphi_r}: A^{1+\epsilon}_{z+\epsilon} \to A^{1+\epsilon}_{z+\epsilon} \text{ is bounded}\}$$

Throughout this paper, the letter *C* will denote constants and may differ from one occurrence to the other. The notation $A \lesssim B$ means that there is a positive constant C such that $A \leq CB$. The The notation $A \approx B$ means $A \lesssim B$ and $B \lesssim A$. The researcher formulated and shows several auxiliary results which will be used in the proofs of main results see [21].

**some Lemmas**

**Lemma 1.** Assume that $(z+\epsilon) \in \widehat{\mathcal{D}}, 0 \leq \epsilon \leq 1$ and $(z+\epsilon)(1-\epsilon) = \int_{1-\epsilon}^{1} s(z+\epsilon)(s)\log\frac{s}{1-\epsilon}\,\mathrm{ds}$. Then the following statements hold.

(i) $(z+\epsilon)_* \in (1-\epsilon) \approx \epsilon(\widehat{z+\epsilon})(1-\epsilon)$ as $(1-\epsilon) \to 1$;

(ii) there are $1 < a < b < +\infty$ and $0 \leq \delta < 1$, such that

$$\frac{(z+\epsilon)_*\,(1-\epsilon)}{\epsilon^{\alpha}} \text{ is decreasing on } [\delta, 1] \text{ and } \lim_{\epsilon\to 0}\frac{(z+\epsilon)_*\,(1-\epsilon)}{\epsilon^{\alpha}} = 0; \qquad (1)$$

$$\frac{(z+\epsilon)_*\,(1-\epsilon)}{\epsilon^{(1+\epsilon)}} \text{ is increasing on } [\delta, 1] \text{ and } \lim_{\epsilon\to 0}\frac{(z+\epsilon)_*\,(1-\epsilon)}{\epsilon^{(1+\epsilon)}} = \infty; \qquad (2)$$

(iii) $(z+\epsilon)_*\,(1-\epsilon)$ is decreasing on $[\delta, 1]$ and $\lim_{\epsilon\to 0}(z+\epsilon)_*\,(1-\epsilon) = 0$.

Proof. By [13, Lemmas A and 9] and [1,19] in [11], (i) and (ii) hold. (iii) follows by (ii) and $(z+\epsilon)_*\,(1-\epsilon) = \frac{(z+\epsilon)_*\,(1-\epsilon)}{\epsilon^{\alpha}}\,\epsilon^{\alpha}$.

**Remark 1.** The researcher observes that $z = 0$ is the logarithmic singular point of $(z+\epsilon)_*$. SO, for any fixed $r_0 \in (0,1)$, we have $(z+\epsilon)_*\,(1-\epsilon) \approx \epsilon(\widehat{z+\epsilon})(1+\epsilon)$ for $r_0 \leq \mathrm{r} < 1$. For simplicity, suppose $(z+\epsilon)_*$ and $\widehat{z+\epsilon}$ radial, that is, $(z+\epsilon)_*(z) = (z+\epsilon)_*(|z|)$ and $\mathrm{z}\widehat{+\epsilon}(z) = \mathrm{z}\widehat{+\epsilon}(|z|)$ for all $z \in \mathbb{D}$.

Suppose $\mathbb{T}$ is the boundary of $\mathbb{D}$ and $\mathrm{I} \subset \mathbb{T}$ is an interval. The Carleson square $S(I)$ can be defined as

$$S(I) = \{(1-\epsilon)e^{it}: e^{it} \in I, |I| \leq \epsilon < -1\},$$

where $|I|$ denotes the Lebesgue measure of $I$. For convenience, for each $a \in \mathbb{D} \setminus \{0\}$, the researcher defines

$$I_{1+\epsilon} = \left\{e^{i\theta}: \left|\arg((1+\epsilon)e^{-i\theta})\right| \leq \frac{1-|1+\epsilon|}{2}\right\},$$

and denote $S(1+\epsilon) = S(I_{(1+\epsilon)})$. By (26) in [10], when $(z+\epsilon) \in \widehat{\mathcal{D}}$, the researcher has

$$z + \epsilon(S(1+\epsilon)) \approx (z+\epsilon)_*(1+\epsilon), \text{ for all } (1+\epsilon) \in \mathbb{D} \text{ and } |1+\epsilon| \geq \frac{1}{2}. \quad (3)$$

The following lemma is a straight result of [10, Lemma 3.1](or [11, Lemma 2.4])

**Lemma 2.** Suppose $(z+\epsilon) \in \widehat{\mathcal{D}}$ and $-1 < \epsilon < \infty$. There exists $\gamma_0 > 0, if\ \gamma > \gamma_0$, the researcher has $\left|F_{(1+\epsilon)^2,(-\frac{1}{2}+3\epsilon)}(z)\right| \approx \frac{1}{(z+\epsilon)S(1+\epsilon))^{\frac{1}{1+\epsilon}}} \left\|F_{(1+\epsilon)^2,(-\frac{1}{2}+3\epsilon)}\right\|_{A^{1+\epsilon}_{z+\epsilon}} \approx 1$, when $(1+\epsilon) \in \mathbb{D}, z \in S(1+\epsilon)$, and

$$\lim_{|1+\epsilon|\to 1} \sup_{|z|\leq 1-\epsilon} \left|F_{(1+\epsilon)^2,(-\frac{1}{2}+3\epsilon)}(z)\right| = 0, \text{ when } 1 < \epsilon < -1.$$

Here and hence forth,

$$F_{(1+\epsilon)^2,(-\frac{1}{2}+3\epsilon)}(z) = \left(\frac{1-|1+\epsilon|^2}{1-(\overline{z+\epsilon})z}\right)^{\frac{1}{2}+3\epsilon} \frac{1}{(z+\epsilon(S(1+\epsilon))^{\frac{1}{1+\epsilon}}}.$$

For simplicity, the researcher always assume that $(-\frac{1}{2}+3\epsilon)$ is large enough so that Lemma 2 holds when mention the function $F_{(1+\epsilon)^2,(-\frac{1}{2}+3\epsilon)}$.

For a given Banach space $X$ of analytic fuuctions on $\mathbb{D}$, appositive Borel measure $\mu$ on $\mathbb{D}$ is called a $q-$ Carleson measure for $X$, if the identity operator $Id: X \to L^{1+2\epsilon}(z)$ is bounded. By [10, Theorem 3.3], when $(z+\epsilon) \in \widehat{\mathcal{D}}$, a Borel measure $\mu$ on $\mathbb{D}$ is a $q-$ Carleson measure for $A^{1+\epsilon}_{z+\epsilon}$ if and only if

$$\sup_{(1+\epsilon)\in\mathbb{D}} \frac{\mu(S(1+\epsilon)}{(z+\epsilon)(S(1+\epsilon))^{\frac{1+2\epsilon}{1+\epsilon}}} < \infty.$$

Moreover, $\|Id\|_{A^{1+\epsilon}_{z+\epsilon}\to L^{1+2\epsilon}_{\mu}} \approx \sup_{(1+\epsilon)\in\mathbb{D}} \frac{\mu(S(1+\epsilon)}{(z+\epsilon)(S(1+\epsilon))^{\frac{1+2\epsilon}{1+\epsilon}}}$. the following lemma.

**Lemma 3.** Suppose $(z+\epsilon) \in \widehat{\mathcal{D}}$, $\mu$ is a positive Borel measure $\mu$ $\mathbb{D}$.Let$-1 < \epsilon \leq 2\epsilon < \infty$. For some (equivalently for all) Large enough $(-\frac{1}{2}+3\epsilon)$, the researcher has

$$\|Id\|^{1+2\epsilon}_{A^{1+\epsilon}_{z+\epsilon}\to L^{1+2\epsilon}_{\mu}} \approx \sup_{(1+\epsilon)\in\mathbb{D}} \frac{\mu(S(1+\epsilon)}{(z+\epsilon)(S(1+\epsilon))^{\frac{1+2\epsilon}{1+\epsilon}}} \approx \sup_{(1+\epsilon)\in\mathbb{D}} \int_{S(1+\epsilon)} \left|F_{(1+\epsilon)^2,(-\frac{1}{2}+3\epsilon)}(z)\right|^{1+2\epsilon} d\mu(z) \approx$$

$$\sup_{(1+\epsilon)\in\mathbb{D}} \int_{\mathbb{D}} \left|F_{(1+\epsilon)^2,(-\frac{1}{2}+3\epsilon)}(z)\right|^{1+2\epsilon} d\mu(z).$$

Here $Id$ is the identity operator.

Proof. By Lemma 2, the researcher has

$$\frac{\mu(S(1+\epsilon)}{(z+\epsilon)(S(1+\epsilon))^{\frac{1+2\epsilon}{1+\epsilon}}} \approx \int_{S(1+\epsilon)} \left|F_{(1+\epsilon)^2,(-\frac{1}{2}+3\epsilon)}(z)\right|^{1+2\epsilon} d\mu(z) \leq$$

$$\int_{\mathbb{D}} \left|F_{(1+\epsilon)^2,(-\frac{1}{2}+3\epsilon)}(z)\right|^{1+2\epsilon} d\mu(z), \text{ when } (1+\epsilon) \in \mathbb{D}. \quad (4)$$

**So**,

$$\sup_{(1+\epsilon)\in\mathbb{D}} \frac{\mu(S(1+\epsilon)}{(z+\epsilon)(S(1+\epsilon))^{\frac{1+2\epsilon}{1+\epsilon}}} \approx \sup_{(1+\epsilon)\in\mathbb{D}} \int_{S(1+\epsilon)} \left|F_{(1+\epsilon)^2,(-\frac{1}{2}+3\epsilon)}(z)\right|^{1+2\epsilon} d\mu(z)$$

$$\lesssim \sup_{(1+\epsilon)\in\mathbb{D}} \int_{\mathbb{D}} \left|F_{(1+\epsilon)^2,(-\frac{1}{2}+3\epsilon)}(z)\right|^{1+2\epsilon} d\mu(z).$$

By Lemma 2 and [10, Theorem 3.3](also see [11, Theorem 2.1]), the researcher obtains

$$\int_{\mathbb{D}} \left|F_{(1+\epsilon)^2,(-\frac{1}{2}+3\epsilon)}(z)\right|^{1+2\epsilon} d\mu(z) \lesssim \|Id\|^{1+2\epsilon}_{A^{1+\epsilon}_{z+\epsilon}\to L^{1+2\epsilon}_{\mu}} \approx \sup_{(1+\epsilon)\in\mathbb{D}} \frac{\mu(S(1+\epsilon)}{(z+\epsilon)(S(1+\epsilon))^{\frac{1+2\epsilon}{1+\epsilon}}}.$$

The proof is complete.

**Lemma 4.** Suppose $-1 < \epsilon \leq 2\epsilon < \infty, (z+\epsilon) \in \widehat{\mathcal{D}}$, $\mu$ is a positive Borel measure $on\ \mathbb{D}$, and $-\frac{1}{2}+3\epsilon$ is large enough. Let $-\frac{1}{2} < \epsilon < 0$ and

$$N_{1-\epsilon}^{*} = \underset{\epsilon \to 0}{sup} \int_{\mathbb{D}} \left| F_{(1+\epsilon)^2,(-\frac{1}{2}+3\epsilon)}(z) \right|^{1+2\epsilon} d\mu(z).$$

If $\mu$ is a $q$-Carleson measure for $A_{z+\epsilon}^{1+\epsilon}$, then $\mu_{r=}\mu\ /\ \mathbb{D} \setminus (1-\epsilon)\mathbb{D}$ is also a q-Carleson measure for $A_{z+\epsilon}^{1+\epsilon}$, where $(1-\epsilon)\mathbb{D} = \{z \in \mathbb{D}: |z| < (1-\epsilon)\}$. Moreover, there is a $C > 0$, such that

$$\sup_{(1+\epsilon)\in\mathbb{D}} \frac{\mu(S(1+\epsilon)}{(z+\epsilon)(S(1+\epsilon)))^{\frac{1+2\epsilon}{1+\epsilon}}} \leq CN_{1-\epsilon}^{*}. \qquad (5)$$

Proof. It is obvious that $\mu_{(1+\epsilon)}$ is a $q$-Carleson measure for $A_{z+\epsilon}^{1+\epsilon}$. Let

$$N_{(1-\epsilon)} = \sup_{|1+\epsilon|\leq(1-\epsilon)} \frac{\mu(S(1+\epsilon)}{(z+\epsilon)(S(1+\epsilon)))^{\frac{1+2\epsilon}{1+\epsilon}}}.$$

When $|1+\epsilon| \leq 1-\epsilon$, $\frac{\mu(S(1+\epsilon)}{(z+\epsilon)(S(1+\epsilon)))^{\frac{1+2\epsilon}{1+\epsilon}}} < N_{(1-\epsilon)}$ is obvious. When $|1+\epsilon| < 1-\epsilon$,

letting $k = int\left(\frac{1-|1+\epsilon|}{\epsilon}\right)+1$, there exists $a_1, a_2, \cdots\cdots, a_k \in \mathbb{D}$ such that $S(1+\epsilon) \cap \mathbb{D} \setminus (1-\epsilon)\mathbb{D} \subset \cup_{i=1}^{k} S((1+\epsilon)_i$ and $|(1+\epsilon)_i|=(1+\epsilon)$ for $i = 1,2,\cdots\cdots,k$. By Lemma 1, the researcher has

$$\begin{aligned}
\mu_{(1-\epsilon)}(S(1+\epsilon) &\leq \textstyle\sum_{i=1}^{k} \mu(S(1+\epsilon)_i \leq N_{(1+\epsilon)} \sum_{i=1}^{k} (z+\epsilon)S(1+\epsilon)_i)^{\frac{1+2\epsilon}{1+\epsilon}} \\
&\lesssim N_{(1-\epsilon)} \left(\frac{1+|1+\epsilon|}{\epsilon}+1\right)(z+\epsilon)_*(1-\epsilon)^{\frac{1+2\epsilon}{1+\epsilon}} \\
&\approx N_{(1-\epsilon)} \left(\frac{1+|1+\epsilon|}{\epsilon}+1\right)(\epsilon)^{\frac{1+2\epsilon}{1+\epsilon}} \widehat{z+\epsilon}\,(1-\epsilon)^{\frac{1+2\epsilon}{1+\epsilon}} \\
&\leq N_{(1-\epsilon)} \left(\left(\frac{\epsilon}{1-|1+\epsilon|}\right)^{\frac{3\epsilon}{1-\epsilon}} + \left(\frac{\epsilon}{1-|1-\epsilon|}\right)^{\frac{1+2\epsilon}{1+\epsilon}}\right)(1-|1+\epsilon)|^{\frac{1+2\epsilon}{1+\epsilon}} \widehat{z+\epsilon}(1+\epsilon)^{\frac{1+2\epsilon}{1+\epsilon}} \\
&\lesssim N_{(1-\epsilon)}(z+\epsilon)(S(1+\epsilon))^{\frac{1+2\epsilon}{1+\epsilon}}
\end{aligned}$$

So, there exists $C > 0$, such that

$$\frac{\mu_{(1+\epsilon)}(S(1+\epsilon))}{z+\epsilon(S(1+\epsilon))^{\frac{1+2\epsilon}{1+\epsilon}}} \leq CN_{(1-\epsilon)}.$$

By(4), the researcher has $N_{(1-\epsilon)} \lesssim N_{(1-\epsilon)}^{*}$. Therefore, (5) holds. The proof is complete.

The following lemma can be proved in standard way (see, for example, Theorem 3.11 in [3]).

**Lemma 5**.Suppose $-1 < \epsilon, \epsilon > 0, (z+\epsilon) \in \widehat{\mathcal{D}}$ and $\mu$ is a positive Borel measure. If $T: A_{z+\epsilon}^{1+\epsilon} \to L_{\mu}^{1+2\epsilon}$ is linear and bounded, then $T$ is compact if and only if whenever $\{f_k^2\}$ is bounded in $A_{z+\epsilon}^{1+\epsilon}$ and $f_k^2 \to 0$ uniformly on compact subsets of $\mathbb{D}, \lim_{k\to 0}\|f_k^2\|_{L_{\mu}^{1+2\epsilon}} = 0$. The following lemma can be found in [18]without a proof. For the benefits of the readers, will prove it.

**Lemma 6**. Let $\varphi_r$ be an analytic self-map of $\mathbb{D}$. Then $\varphi_r$ is a finite Blasschke product if and only if $\lim_{|z+\epsilon|\to 1}|\varphi_r(z+\epsilon)|=1$.

Proof. The sufficiency of the statement is obvious. Next the researcher shows the necessity.

Suppose $\lim_{|z+\epsilon|\to 1}|\varphi_r(z+\epsilon)|=1$. Let $(F-C) \subset \mathbb{D}$ be compact. Then there exists $0 < r < 1$ such that $(-C) \subset (1-\epsilon)\overline{\mathbb{D}}$, where $(1-\epsilon)\overline{\mathbb{D}} = \{z \in \mathbb{D}: |z| \leq (1-\epsilon)\}$. Since $\lim_{|z+\epsilon|\to 1}|\varphi_r(z+\epsilon)|=1$, there is a $0 < t < 1$ such that for all $|z| > t$, the researcher has $|\varphi_r(z)| > (1-\epsilon)$. Therefore, $\varphi_r^{-1}(F-C) \subset t\overline{\mathbb{D}}$. By the continuity of $\varphi_{r,}\varphi_r^{-1}(F-C)$ is closed. So, $\varphi_r^{-1}(F-C)$ is compact. By the subsection 7.1.3 of [17], $\varphi_r$ is proper. By the subsection 7.3.1 of [17] , $\varphi_r$ is a finite Blaschke product. The proof is complete.

**Main Results and proofs**

**Theorem 1.** Assume $(z+\epsilon)\in\mathcal{D}$, $-1\le\frac{\epsilon}{2}<\epsilon<0$, $(1+\epsilon)\colon\mathbb{D}\to\mathbb{C}$ is a measurable function, $\varphi_r$ is an analytic self-map of $\mathcal{D}$, and $\mu$ is a positive Borel measure on $\mathbb{D}$. Then

$$\left\|(1+\epsilon)C_{\varphi_r}\right\|^{1+2\epsilon}_{A^{1+\epsilon}_{z+\epsilon}\to L^{1+2\epsilon}_{\mu}}\ \approx\ \sup_{(1+\epsilon)\in\mathbb{D}}\int_{\mathbb{D}}\left|F_{(1+\epsilon)^2,(-\frac{1}{2}+3\epsilon)}(\varphi_r(z)\right|^{1+2\epsilon}\ \|(1+(z)\|^{1+2\epsilon}\quad d\mu(z).$$

Proof. By lemma 2, the researcher has

$$\sup_{(1+\epsilon)\in\mathbb{D}}\int_{\mathbb{D}}\left|F_{(1+\epsilon)^2,(-\frac{1}{2}+3\epsilon)}(\varphi_r(z)\right|^{1+2\epsilon}\ \|(1+\epsilon)(z)\|^{1+2\epsilon}\quad d\mu(z)\lesssim\left\|(1+\epsilon)C_{\varphi_r}\right\|^{1+2\epsilon}_{A^{1+\epsilon}_{z+\epsilon}\to L^{1+2\epsilon}_{\mu}}.$$

Let $v(F-C)=\int_{\varphi^{-1}(E)}|(1+\epsilon)(z)|^{1+2\epsilon}\quad d\mu(z)$ for all Borel set $(F-C)$. For all $f^2\in A^{1+\epsilon}_{z+\epsilon,}$ letting $z+\epsilon=\varphi_r(z)$, by lemma 3 the researcher has

$$\left\|1+\epsilon)C_{\varphi_r}f^2\right\|^{1+2\epsilon}_{L^{1+2\epsilon}_{\mu}}=\int_{\mathbb{D}}|f^2(\varphi(z)|^{1+2\epsilon}\,|(1+\epsilon)(z)|^{1+2\epsilon}\,d\mu(z)\qquad(6)$$

$$=\int_{\mathbb{D}}|f^2(z+\epsilon)|^{1+2\epsilon}\,dv(z+\epsilon)$$

$$=\ \|f^2\|^{1+2\epsilon}_{A^{1+2\epsilon}_{v}}\le\|Id\|^{1+2\epsilon}_{A^{1+\epsilon}_{z+\epsilon}\to L^{1+2\epsilon}_{v}}\,\|f^2\|^{1+2\epsilon}_{A^{1+\epsilon}_{z+\epsilon}}.$$

$$\lesssim\ \|f^2\|^{1+2\epsilon}_{A^{1+\epsilon}_{z+\epsilon}}\sup_{(1+\epsilon)\in\mathbb{D}}\int_{\mathbb{D}}\left|F_{(1+\epsilon)^2,(-\frac{1}{2}+3\epsilon)}(z+\epsilon)\right|^{1+2\epsilon}\ dv(z+\epsilon).$$

Making the changing of variable $z+\epsilon=\varphi_r(z)$, the researcher has

$$\int_{\mathbb{D}}\left|F_{(1+\epsilon)^2,(-\frac{1}{2}+3\epsilon)}(z+\epsilon)\right|^{1+2\epsilon}=\int_{\mathbb{D}}dv(z+\epsilon)=\left|F_{(1+\epsilon)^2,(-\frac{1}{2}+3\epsilon)}(\varphi_r(z)\right|^{1+2\epsilon}\ \|(1+\epsilon)(z)\|^{1+2\epsilon}\quad d\mu(z).$$

The proof is complete.

Let $X$ and $Y$ be Banach spaces. Recall that the essential norm of linear operator $T\colon X\to Y$ is defined by

$$\|T\|_{e,X\to Y}=\inf\{\|T-k\|_{X\to Y}\colon K\text{ is compact from }X\ to\ Y\}.$$

Obviously $T\colon X\to Y$ is compact if and only if $\|T\|_{e,X\to Y}=0$.

**Theorem 2.** Assume $(z+\epsilon)\in\widehat{\mathcal{D}},-1\le\frac{\epsilon}{2}<\epsilon<0,(1+\epsilon)\colon\mathbb{D}\to\mathbb{C}$ is a measurable function,$\varphi_r$ is an analytic self-map of $\mathcal{D}$, and $\mu$ is a positive Borel measure on $\mathbb{D}$. If $(1+\epsilon)C_{\varphi_r}\colon A^{1+\epsilon}_{z+\epsilon}\to A^{1+2\epsilon}_{\mu}$ is bounded, then

$$\left\|(1+\epsilon)C_{\varphi_r}\right\|^{1+2\epsilon}_{e,A^{1+\epsilon}_{z+\epsilon}\to A^{1+2\epsilon}_{\mu}}\approx\ \lim_{|1+\epsilon|\to1}\sup\ \int_{\mathbb{D}}\left|F_{(1+\epsilon)^2,(-\frac{1}{2}+3\epsilon)}(\varphi_r(z)\right|^{1+2\epsilon}\ \|(1+\epsilon)(z)\|^{1+2\epsilon}\quad d\mu(z).$$

Proof. Since $(1+\epsilon)C_{\varphi_r}\colon A^{1+\epsilon}_{z+\epsilon}\to A^{1+2\epsilon}_{\mu}$ is bounded, $(1+\epsilon)\in L^{1+2\epsilon}_{\mu}$ and $\|1+\epsilon\|_{L^{1+2\epsilon}_{\mu}}\le\|(1+\epsilon)C\varphi_r\|_{A^{1+2\epsilon}_{z+\epsilon}\to L^{1+2\epsilon}_{\mu}}$. **Upper estimate of** $\left\|(1+\epsilon)C_{\varphi_r}\right\|^{1+2\epsilon}_{e,A^{1+\epsilon}_{z+\epsilon}\to A^{1+2\epsilon}_{\mu}}$. Suppose $f(z)=\sum_{k=0}^{\infty}\hat{f_k^2}\,z^k\ \in H(\mathbb{D})$. For $\epsilon\in\mathbb{N}$, let

$$K_\epsilon f^2(z)=\begin{cases}\sum_{k=0}^{\epsilon}\hat{f_k^2}\,z^k\,, & \epsilon>0;\\ \sum_{k=0}^{\epsilon}\left(1-\frac{k}{\epsilon+1}\right)\hat{f_k^2}\,z^k\,, & \epsilon=0,\end{cases}$$

and $R_\epsilon=Id-K_\epsilon$.

By[19, proposition 1 and Corollary 3], when $0<\epsilon<\infty$, $K_\epsilon$ is bounded uniformly on $H^{1+\epsilon}$. By [6], $\|K_\epsilon\|_{H^1\to H^1}\le1$. So, when $\epsilon\ge0$, there is a $C=C(1+\epsilon)$ such that

$$\|K_\epsilon f^2\|^{1+\epsilon}_{A^{1+\epsilon}_{Z+\epsilon}}\le C\ \int_0^1(z+\epsilon)(1+\epsilon)d(1+\epsilon)\ \int_0^{2\pi}\left\|f^2(1+\epsilon)e^{i\theta}\right\|^{1+\epsilon}d\theta\le C\|f^2\|^{1+\epsilon}_{A^{1+\epsilon}_{Z+\epsilon}},$$

$$\|R_\epsilon\|_{A^{1+\epsilon}_{Z+\epsilon}\to A^{1+\epsilon}_{Z+\epsilon}}=\|Id-K_\epsilon\|_{A^{1+\epsilon}_{Z+\epsilon}\to A^{1+\epsilon}_{Z+\epsilon}}\le1+\|K_\epsilon\|_{A^{1+\epsilon}_{Z+\epsilon}\to A^{1+\epsilon}_{Z+\epsilon}}\le C+1.\qquad(7)$$

By Lemma 5 and Cauchy's estimate, $f^2\to\hat{f_k^2}\,z^k$ is compact on $A^{1+\epsilon}_{z+\epsilon}$. Therefore, $K_\epsilon\colon A^{1+\epsilon}_{z+\epsilon}\to A^{1+\epsilon}_{z+\epsilon}$ is compact. So, the researcher has

$$\left\|(1+\epsilon)C_{\varphi_r}\right\|^{1+2\epsilon}_{e,L^{1+\epsilon}_{z+\epsilon}\to A^{1+2\epsilon}_{\mu}} = \left\|(1+\epsilon)C_{\varphi_r}(K_\epsilon+R_\epsilon)\right\|^{1+2\epsilon}_{e,A^{1+\epsilon}_{z+\epsilon}\to L^{1+2\epsilon}_{\mu}}$$

$$\leq \left\|(1+\epsilon)C_{\varphi_r}R_\epsilon\right\|^{1+2\epsilon}_{e,A^{1+\epsilon}_{z+\epsilon}\to L^{1+2\epsilon}_{\mu}}$$

For any fixed $0<r<1$, by (6) the researcher has

$$\left\|(1+\epsilon)C_{\varphi_r}R_\epsilon f^2\right\|^{1+\epsilon}_{L^{1+2\epsilon}_{\mu}} = \int_{\mathbb{D}\setminus(1-\epsilon)\mathbb{D}}|R_\epsilon f^2(z+\epsilon)|^{1+2\epsilon}\,d\upsilon(z+\epsilon)+\int_{(1-\epsilon)\mathbb{D}}|R_\epsilon f^2(z+\epsilon)|^{1+2\epsilon}\,d\upsilon(z+\epsilon), \quad (9)$$

Where $\upsilon$ is defined in the proof of the proof of Theorem 1, that is, for every Borel set $(F-C)\in\mathbb{D}$. $\upsilon(F-C)=\int_{\varphi^{-1}(F-C)}|(1+\epsilon)(z)|^{1+2\epsilon}\,d\mu(z)$.

**Let** $(z+\epsilon)_{\epsilon\epsilon}=\int_0^1(1+\epsilon)^\epsilon(z+\epsilon)(1+\epsilon)d(1+\epsilon)$. Since $B_z^{z+\epsilon}(\xi)=\sum_{k=0}^{\infty}\frac{(\xi z)^\epsilon}{2(z+\epsilon)_{2\epsilon+1}}$ is the reproducing Kernel of $A^{1+\epsilon}_{z+\epsilon}$ (see [14,16] for example), the researcher has

$$|R_\epsilon f^2(z+\epsilon)|=\left|\langle R_\epsilon f^2,B^\omega_{Z+\epsilon}\rangle_{A^2_{z+\epsilon}}\right|=\left|\langle f^2,R_\epsilon B^\omega_{Z+\epsilon}\rangle_{A^2_{z+\epsilon}}\right|\lesssim\|f^2\|_{A^{1+\epsilon}_{z+\epsilon}}\|R_\epsilon B^\omega_{z+\epsilon}\|_{H^\infty}.$$

Here $\langle\cdot,\cdot\rangle_{A^2_{z+\epsilon}}$ is the inner product induced by $\|\cdot\|_{A^2_{z+\epsilon}}$ and $\|\cdot\|_{H^\infty}$ is the norm of the bounded analysis function space on $\mathbb{D}$. When $|z+\epsilon|\leq(1-\epsilon)$, the researcher has

$$\|R_\epsilon B^\omega_{z+\epsilon}\|_{H^\infty}\leq\frac{1}{\epsilon}\sum_{k=1}^{\infty}\frac{k(1-\epsilon)^{k-1}}{2\omega_{2k+1}}+\sum_{k=\epsilon+1}^{\infty}\frac{(1-\epsilon)^k}{2\omega_{2k+1}}.$$

By [16, Lemma 6], $\sum_{k=1}^{\infty}\frac{k(1-\epsilon)^{k-1}}{2\omega_{2k+1}}$ is convergent and $\lim_{n\to\infty}\sum_{k=\epsilon+1}^{\infty}\frac{(1-\epsilon)^k}{2\omega_{2k+1}}=0$. So, for all $\epsilon>0$, there is a $N=N(\varepsilon,z+\epsilon,1-\epsilon)$, such that

$$|R_\epsilon B^\omega_{z+\epsilon}|<\varepsilon, \text{ for all } |z+\epsilon|\leq(1-\epsilon) \text{ and } \epsilon>\mathrm{N}.$$

So, for all $\epsilon>\mathrm{N}$,

$$\int_{(1-\epsilon)\mathbb{D}}|R_\epsilon f^2(z+\epsilon)|^{1+2\epsilon}\,d\upsilon(z+\epsilon)\leq\epsilon^{1+2\epsilon}\|1+\epsilon\|^{1+2\epsilon}_{L^{1+2\epsilon}_{\mu}}\|f^2\|^{1+2\epsilon}_{A^{1+\epsilon}_{z+\epsilon}}. \quad (10)$$

Let $\upsilon_{(1-\epsilon)}=\upsilon\setminus_{\mathbb{D}\setminus(1-\epsilon)\mathbb{D}}$. By (7), Lemmas 3 and 4, the researcher has

$$\int_{\mathbb{D}\setminus(1-\epsilon)\mathbb{D}}|R_\epsilon f^2(z+\epsilon)|^{1+2\epsilon}\,d\upsilon(z+\epsilon)=\int_{\mathbb{D}}|R_\epsilon f^2(z+\epsilon)|^{1+2\epsilon}\,d\upsilon_{(1-\epsilon)}(z+\epsilon)$$

$$\lesssim\|R_\epsilon f^2\|^{1+2\epsilon}_{A^{1+\epsilon}_{z+\epsilon}}\sup_{1+\epsilon}\frac{\upsilon_{(1-\epsilon)}(S(1+\epsilon)}{\omega(S(1+\epsilon))^{\frac{1+2\epsilon}{1+\epsilon}}}$$

$$\lesssim\|R_\epsilon f^2\|^{1+2\epsilon}_{A^{1+\epsilon}_{z+\epsilon}}\sup_{\epsilon>0}\int_{\mathbb{D}}\left|F_{(1+\epsilon)^2,(-\frac{1}{2}+3\epsilon)}(\varphi_r(z)\right|^{1+2\epsilon}\|(1+\epsilon)(z)\|^{1+2\epsilon}\,d\mu(z).$$

$$\lesssim\|f^2\|^{1+2\epsilon}_{A^{1+\epsilon}_{z+\epsilon}}\sup_{\epsilon>0}\int_{\mathbb{D}}\left|F_{(1+\epsilon)^2,(-\frac{1}{2}+3\epsilon)}(\varphi_r(z)\right|^{1+2\epsilon}\|(1+\epsilon)(z)\|^{1+2\epsilon}\,d\mu(z). \quad (11)$$

Letting $\epsilon\to0$, by (8)-(11), the researcher gets

$$\left\|(1+\epsilon)C_{\varphi_r}\right\|^{1+2\epsilon}_{e,L^{1+\epsilon}_{z+\epsilon}\to A^{1+2\epsilon}_{\mu}}\lesssim$$

$$\int_{\mathbb{D}}\left|F_{(1+\epsilon)^2,(-\frac{1}{2}+3\epsilon)}(\varphi_r(z)\right|^{1+2\epsilon}\|(1+\epsilon)(z)\|^{1+2\epsilon}\,d\mu(z)+\varepsilon^{1+2\epsilon}\|1+\epsilon\|^{1+2\epsilon}_{L^{1+2\epsilon}_{\mu}}.$$

Since $\varepsilon$ is arbitrary, by letting $\epsilon\to0$, the researcher obtains

$$\left\|(1+\epsilon)C_{\varphi_r}\right\|^{1+2\epsilon}_{e,L^{1+\epsilon}_{z+\epsilon}\to A^{1+2\epsilon}_{\mu}}\lesssim\limsup_{\epsilon\to0}\int_{\mathbb{D}}\left|F_{(1+\epsilon)^2,(-\frac{1}{2}+3\epsilon)}(\varphi_r(z)\right|^{1+2\epsilon}\|(1+\epsilon)(z)\|^{1+2\epsilon}\,d\mu(z).$$

**Lower estimate of** $\left\|(1+\epsilon)C_{\varphi_r}\right\|^{1+2\epsilon}_{e,L^{1+\epsilon}_{z+\epsilon}\to A^{1+2\epsilon}_{\mu}}$.

Assume that $K:A^{1+\epsilon}_{z+\epsilon}\to L^{1+2\epsilon}_{\mu}$ is compact. By Lemmas 2 and 5. $\lim_{\epsilon\to0}\left\|KF_{(1+\epsilon)^2,(-\frac{1}{2}+3\epsilon)}\right\|_{A^{1+2\epsilon}_{\mu}}=0$ then

$$\left\|(1+\epsilon)C_{\varphi_r}\right\|_{A^{1+\epsilon}_{z+\epsilon}\to L^{1+2\epsilon}_{\mu}}\gtrsim\limsup_{\epsilon\to0}\left\|((1+\epsilon)C_{\varphi_r}-K)F_{(1+\epsilon)^2,(-\frac{1}{2}+3\epsilon)})\right\|_{L^{1+2\epsilon}_{\mu}}$$

$$\geq \limsup_{\epsilon \to 0} \left\|(1+\epsilon)\varphi_r F_{(1+\epsilon)^2,(-\frac{1}{2}+3\epsilon)}\right\|_{L_\mu^{1+2\epsilon}} - \limsup_{\epsilon \to 0} \left\|KF_{(1+\epsilon)^2,(-\frac{1}{2}+3\epsilon)}\right\|_{L_\mu^{1+2\epsilon}}$$

$$= \limsup_{\epsilon \to 0} \left\|(1+\epsilon)\varphi_r F_{(1+\epsilon)^2,(-\frac{1}{2}+3\epsilon)}\right\|_{L_\mu^{1+2\epsilon.}}$$

Therefore, the researcher gets

$$\left\|(1+\epsilon)C_{\varphi_r}\right\|^{1+2\epsilon}_{e,L^{1+\epsilon}_{z+\epsilon}\to A^{1+2\epsilon}_\mu} \gtrsim \limsup_{\epsilon \to 0} \int_{\mathbb{D}} \left|F_{(1+\epsilon)^2,(-\frac{1}{2}+3\epsilon)}(\varphi_r(z)\right|^{1+2\epsilon} \|(1+\epsilon)(z)\|^{1+2\epsilon}\, d\mu(z).$$

as desired. The proof is complete.

**Theorem 3.** Assume$(z+\epsilon) \in \widehat{\mathcal{D}}, -1 \leq \frac{\epsilon}{2} < \epsilon < 0, (1+\epsilon)\colon \mathbb{D} \to \mathbb{C}$ is a measurable function,$\varphi_r$ is an analytic self-map of $\mathcal{D}$, and $\mu$ is a positive Borel measure on $\mathbb{D}$. Then the following statements are equivalent.

(i) $(1+\epsilon)\varphi_r\colon A^{1+\epsilon}_{z+\epsilon} \to L^{1+2\epsilon}_\mu$ is bounded;

(ii) $(1+\epsilon)\varphi_r\colon A^{1+\epsilon}_{z+\epsilon} \to L^{1+2\epsilon}_\mu$ is bounded;

(iii) $\Psi^{-\frac{1}{2}+3\epsilon}_{(1+\epsilon)\varphi_r}(1+\epsilon) \in L^{-\frac{1}{\epsilon}-1}_{z+\epsilon}$ for all $-\frac{1}{2}+3\epsilon$ large enough;

(iv) $\Psi^{-\frac{1}{2}+3\epsilon}_{(1+\epsilon)\varphi_r}(1+\epsilon) \in L^{-\frac{1}{\epsilon}-1}_{z+\epsilon}$ for some $-\frac{1}{2}+3\epsilon$ large enough.

Moreover, if $-\frac{1}{2}+3\epsilon$ is fixed,

$$\left\|(1+\epsilon)C_{\varphi_r}\right\|^{1+2\epsilon}_{L^{1+\epsilon}_{z+\epsilon}\to A^{1+2\epsilon}_\mu} \approx \left\|\Psi_{-\frac{1}{2}+3\epsilon}\right\|_{L^{-\frac{1}{\epsilon}-1}_{z+\epsilon}}. \qquad (12)$$

**Here**,

$$\Psi^{-\frac{1}{2}+3\epsilon}_{(1+\epsilon)\varphi_r}(1+\epsilon) = \int_{\mathbb{D}} \left|F_{(1+\epsilon)^2,(-\frac{1}{2}+3\epsilon)}(\varphi_r(z)\right|^{1+2\epsilon} \|(1+\epsilon)(z)\|^{1+2\epsilon}\, d\mu(z).$$

Proof. Let $\upsilon(F-C) = \int_{\varphi^{-1}(F-C)} |(1+\epsilon)(z)|^{1+2\epsilon}\ \ d\mu(z)$ for al Borel set $(F-C)$. By (6), the researcher has

$$\|(1+\epsilon)C_r f^2\|^{1+2\epsilon}_{L^{1+2\epsilon}_\mu} = \|f^2\|^{1+2\epsilon}_{L^{1+2\epsilon}_\upsilon}.$$

So, $(1+\epsilon)\varphi_r\colon A^{1+\epsilon}_{z+\epsilon} \to L^{1+2\epsilon}_\mu$ is bounded (compact) if and only if $Id\colon A^{1+\epsilon}_{z+\epsilon} \to L^{1+2\epsilon}_\upsilon$ is bounded (compact). By (26) in [10] and Theorem 3 in [15], the researcher has (i) $\Leftrightarrow$(ii)$\Leftrightarrow$ $(iii)$ and (12). Since $1-|1+\epsilon| \leq |1-\overline{1+\epsilon}z|$ holds for all $(1+\epsilon), z \in \mathbb{D}$, the researcher gets (iii) $\Leftrightarrow$ (iv). The proof is complete.

**Remark 2**. Suppose all of the $(1+\epsilon), (1+2\epsilon), \mu, (z+\epsilon), (1+\epsilon), \varphi_r$ meet the conditions of the Theorem 3 and $\upsilon$ is defined as in the proof of Theorem 3. For all $(1+\epsilon) \in \mathbb{D} \setminus \{0\}$ and $1 < \epsilon < 0$, let

$$\Gamma(1+\epsilon) = \left\{z \in \mathbb{D}\colon |\text{arg}z - \arg(1+\epsilon)| < \frac{1}{2}\left(1 - \frac{|z|}{|1+\epsilon|}\right)\right\}$$

$T(1+\epsilon) = z \in \mathbb{D}\colon \{(1+\epsilon) \in \Gamma(z)\}, \Delta(1+\epsilon), (1-\epsilon) = \left\{z \in \mathbb{D}\colon \left|\frac{(1+\epsilon)-z}{1-\overline{1+\epsilon}z}\right| < 1-\epsilon\right\}$,

and

$$Q(z) = \int_\Gamma\ (z)\frac{d\upsilon(\xi)}{\omega(T(\xi))}, M_\omega(\upsilon(z) \underset{z \in S(1+\epsilon)}{sup} \frac{\upsilon(S(1+\epsilon))}{\omega(S(1+\epsilon))}, \Phi_{(1-\epsilon)}(z) = \int_{\Gamma(z)} \frac{\upsilon(\Delta(1+\epsilon),(1-\epsilon)}{\omega(T(z))}\frac{dA(z)}{(1-|z|)^2}.$$

By[15, Theorem 3], for any fixed $-\frac{1}{2}+3\epsilon$ and $1-\epsilon$, the researcher has

$$\left\|(1+\epsilon)C_{\varphi_r}\right\|^{1+2\epsilon}_{A^{1+\epsilon}_{z+\epsilon}\to L^{1+2\epsilon}_\mu} \approx \|Id\|^{1+2\epsilon}_{A^{1+\epsilon}_{z+\epsilon}\to L^{1+2\epsilon}_\mu} \approx \left\|\Psi_{-\frac{1}{2}+3\epsilon}\right\|_{L^{-\frac{1}{\epsilon}-1}_{z+\epsilon}} \approx \|M_\omega(z)\|_{L^{-\frac{1}{\epsilon}-1}_{z+\epsilon}} \approx \|Q\|_{L^{-\frac{1}{\epsilon}-1}_{z+\epsilon}} \approx$$

$$\left\|\Phi_{(1-\epsilon)}\right\|_{L^{-\frac{1}{\epsilon}-1}_{z+\epsilon}}. \qquad (13)$$

Using (26) in [10], the researcher knows that all the all the $\omega(S(1+\epsilon)$ and $\omega(T(1+\epsilon))$ in (12) and (13) can be exchanged.

**Theorem 4.** Assume $(z+\epsilon) \in \mathcal{D}$, satisfying $\int_0^1 \left(\log\frac{e}{\epsilon}\right)^2\ (z+\epsilon)(1+\epsilon)d(1+\epsilon) < \infty$,

$-1 < \epsilon < \infty, (1+\epsilon)\colon \mathbb{D} \to \mathbb{C}$ is a measurable function, $\varphi_r$ is an analytic self- map of $\mathbb{D}$. if $(1+\epsilon)\varphi_r\colon A^2_{z+\epsilon} \to A^2_{z+\epsilon}$ is compact, then $(1+\epsilon)\varphi_r \in S_{(1+\epsilon)}(A^2_{z+\epsilon})$ if and only if

$$\int_{\mathbb{D}} \left(\frac{\sigma(\Delta(z(1-\epsilon)}{(z+\epsilon)_*(z)}\right)^{\frac{1+\epsilon}{2}} \frac{dA(z)}{1-2|z|^2+|z|^4} < \infty$$

For some (equivalently for all) $1 < \epsilon < 0$. Moreover, the researcher has

$$\left|(1+\epsilon)C_{\varphi_r}\right|^{1+\epsilon}_{1+\epsilon} \approx \int_{\mathbb{D}} \left(\frac{\sigma(\Delta(z(1-\epsilon)}{(z+\epsilon)_*(z)}\right)^{\frac{1+\epsilon}{2}} \frac{dA(z)}{1-2|z|^2+|z|^4}.$$

Here $\sigma(F-C) = \int_{\varphi^{-1}(F-C)} |(1+\epsilon)(z)|^2 \ (z+\epsilon)\ dA(z)$ for all Borel set $(F-C) \subset \mathbb{D}$, and $|\cdot|_{1+\epsilon}$ is the norm of p- Schatten class of $A^2_{z+\epsilon}$.

Proof. For all $f^2, g^2 \in A^2_{z+\epsilon}$, the researcher has

$$\langle (1+\epsilon)C\varphi_r)^* 1+\epsilon)\varphi_r f^2, g^2\rangle_{A^2_{z+\epsilon}} = \langle (1+\epsilon)C_{\varphi_r} f^2, (1+\epsilon)C_{\varphi_r} g^2\rangle_{A^2_{z+\epsilon}} = \int_{\mathbb{D}} f^2\varphi(z)\overline{g^2\varphi(z)}\,|(1+\epsilon)(z)|^2(z+\epsilon)(z)dA(z) = \int_{\mathbb{D}} f^2(\zeta)\overline{g^2(\xi)}\ d\sigma(\xi) \quad (14)$$

Suppose $B^{z+\epsilon}_z(\xi)$ is the reproducing Kernel of $A^2_{z+\epsilon}$, that is,

$$f^2(z) = \langle f^2, B^{z+\epsilon}_z\rangle_{A^2_\omega} = \int_{\mathbb{D}} f^2(\xi)\overline{B^{z+\epsilon}_z(\xi)}\ (z+\epsilon)(\xi)dA(\xi).$$

Consider the Toeplitz operator,

$$T_\sigma f^2(z) = \int_{\mathbb{D}} f^2(\eta)\,\overline{B^{z+\epsilon}_z(\eta)}d\sigma(\eta).$$

Since $z+\epsilon$ is radial, by [11, section 4.1], polynomials are dense in $A^{1+\epsilon}_{z+\epsilon}$ for all $-1 < \epsilon < \infty$. So if $f^2, g^2 \in A^2_{z+\epsilon}$, there are two polynomial sequences $\{f^2_{(1+\epsilon)}\}^{\infty}_{\epsilon=0}$ such that

$$\lim_{\epsilon \to \infty} \left\|f^2 - f^2_{(1+\epsilon)}\right\|_{A^2_{z+\epsilon}} = 0, \text{ and } \lim_{\epsilon \to \infty} \left\|g^2 - g^2_{(1+\epsilon)}\right\|_{A^2_{z+\epsilon}} = 0.$$

Since $(1+\epsilon)C_{\varphi_r}\colon A^2_{z+\epsilon} \to A^2_{z+\epsilon}$ is compact, by Theorem 1 and lemma 3, the researcher has

$$\left\|(1+\epsilon)C_{\varphi_r}\right\|^2_{A^2_{z+\epsilon}\to A^2_{z+\epsilon}} \approx \sup_{(1+\epsilon)\in\mathbb{D}} \int_{\mathbb{D}} \left|F_{(1+\epsilon)^2,(-\frac{1}{2}+3\epsilon)}(z)\right|^2 d\sigma(z) \approx$$

$\sup_{(1+\epsilon)\in\mathbb{D}} \frac{\sigma(S(1+\epsilon))}{\omega(S(1+\epsilon))} < \infty.$

Then, $Id\colon A^2_\omega \to A^2_\sigma$ and $Id\colon A^1_\omega \to A^1_\sigma$ are bounded by Lemma 3. So, the researcher has

$$\lim_{k\to\infty} \|g^2 - g^2_k\|_{A^2_\sigma} = 0.$$

For any $h^2 \in H^\infty \subset A^1_{z+\epsilon,}$ letting $M = sup_{z\in\mathbb{D}}|h(z)|$, by [16] (see also [14, Theorem 1] ), there exist a $C = C(h^2, 1+\epsilon, \varphi_r, z+\epsilon)$ such that

$$|T_\sigma h^2(z)| \le M \int_{\mathbb{D}} |B^{Z+\epsilon}_z(\eta)|\, d\sigma(\eta) \le M, \|Id\|_{A^1_{z+\epsilon}\to A^1_{z+\epsilon}} \|B^{z+\epsilon}_z\|_{A^1_{z+\epsilon}} \le C \log\frac{e}{1-|z|}.$$

**Therefore**,

$$\|T_\sigma h^2\|^2_{A^2_{z+\epsilon}} \le \int_{\mathbb{D}} \left(\log\frac{e}{1-|z|}\right)^2 (z+\epsilon)\,(z)dA(z) < \infty.$$

That is say, $T_\sigma h^2 \in A^2_{z+\epsilon}$.

Since $g^2_{1+\epsilon} \in A^2_{z+\epsilon} = (A^2_{z+\epsilon})^*$, for any $(1+\epsilon) \in \mathbb{N}$, by Lemma 11 of [16],

$$\langle T_\sigma f^2_{(1+\epsilon)}, g^2\rangle_{A^2_{z+\epsilon}} = \lim_{k\to\infty} \langle T_\sigma f^2_{(1+\epsilon)}, g^2_k\rangle_{A^2_{z+\epsilon}} = \lim_{k\to\infty} \langle f^2_{(1+\epsilon)}, g^2_k\rangle_{A^2_{z+\epsilon}} = \int_{\mathbb{D}} f^2_{(1+\epsilon)}(\eta)\overline{g^2(\eta)}\ d\sigma(\eta). \quad (15)$$

Since $g^2$ is arbitrary, by (14) and (15), the researcher has

$$T_\sigma f_{(1+\epsilon)} = (1+\epsilon)C_{\varphi_r})^*(1+\epsilon)C_{\varphi_r})f_{(1+\epsilon)}. \quad (16)$$

For any fixed $z_0 = \mathbb{D}$, when $|z-z_0| < \frac{1+|z_0|}{2}$, by Hölder inequality and [16, Lemma 6], the researcher has

$$\left|T_\sigma f^2(z) - T_\sigma f^2_{(1+\epsilon)}(z)\right| \le \int_{\mathbb{D}} \left|f^2(\eta) - f^2_{(1+\epsilon)}(\eta)\right| \left|\overline{B^{z+\epsilon}_z(\eta)}\right| d\sigma(\eta)\eta$$

$$\lesssim \sup_{|z| < (|z_0|+1)/2} \frac{1}{\omega(S(z))} \sqrt{\sigma(\mathbb{D})} \|f^2 - f_n^2\|_{A^2_\sigma}.$$

So, the researcher obtains $\lim_{\epsilon \to 0} T_\mu f^2_{1+\epsilon}(z) = T_\mu f^2(z)$ for all $z \in \mathbb{D}$. Using (16), the researcher has

$$T_\sigma = (1+\epsilon) C\varphi_r)^* (1+\epsilon)\varphi_r.$$

By [20, Theorem 1.26], when $\epsilon > 0$, $(1+\epsilon)C_{\varphi_r} \in S_{(1+\epsilon)}(A^2_{z+\epsilon})$ if and only if $((1+\epsilon)C_{\varphi_r})^*(1+\epsilon)C_{\varphi_r} \in S_{\frac{1+\epsilon}{2}}(A^2_{z+\epsilon})$. By [16, Theorem 3] (or[13, Theorem 1]), $T_\sigma = \in S_{\frac{1+\epsilon}{2}}(A^2_{z+\epsilon})$ if and only if

$$\int_{\mathbb{D}} \left(\frac{\sigma(\Delta(z,1-\epsilon)}{(z+\epsilon)_*(z)}\right)^{\frac{1+\epsilon}{2}} \frac{dA(Z)}{(1-|Z|^2)^2} < \infty,$$

For some (equivalently for all) $0 < r < 1$. Moreover,

$$\left|((1+\epsilon)C_{\varphi_r}))\right|^{1+\epsilon}_{1+\epsilon} = |T_\sigma|^{\frac{1+\epsilon}{2}}_{\frac{1+\epsilon}{2}} = \int_{\mathbb{D}} \left(\frac{\sigma(\Delta(z,1-\epsilon)}{(z+\epsilon)_*(z)}\right)^{\frac{1+\epsilon}{2}} \frac{dA(Z)}{(1-|Z|^2)^2}.$$

The proof is complete.

**Theorem 5.** Suppose $0 < \epsilon < \infty$, $(z+\epsilon) \in \mathcal{R}$, $\varphi_r$ is a finite Blaschke product and $(1+\epsilon) \in A^{1+\epsilon}_{z+\epsilon.}$ If $(1+\epsilon)C_{\varphi_r})$ is bounded on $A^{1+\epsilon}_{z+\epsilon,}$ then $(1+\epsilon) \in H^\infty(\mathbb{D})$.

Proof. By[14, corollary 7], the researcher has$(A^{1+\epsilon}_{z+\epsilon})^* \approx A^{(1+\epsilon)'}_{z+\epsilon}$ where $\frac{1}{1+\epsilon} + \frac{1}{(1+\epsilon)'} = 1$. Let $B^{z+\epsilon}_z(\xi)$ be the reproducing kernel of $A^2_{z+\epsilon}$. By [16, Lemma 6], $B^{z+\epsilon}_z \in H^\infty \subset A^{(1+\epsilon)'}_{z+\epsilon}$. Since If $(1+\epsilon)C_{\varphi_r})$ is bounded on $A^{1+\epsilon}_{z+\epsilon,}$ so is $((1+\epsilon)C_{\varphi_r})^*$ on $A^{(1+\epsilon)'}_{z+\epsilon}$. For all $f^2 \in A^{1+\epsilon}_{z+\epsilon,}$ the researcher has $\langle((1+\epsilon)C_{\varphi_r})^* B^{z+\epsilon}_z, f^2\rangle_{A^2_{z+\epsilon}} = \langle B^{z+\epsilon}_Z, (1+\epsilon)C_{\varphi_r} f^2\rangle_{A^2_{z+\epsilon}} = \overline{(1+\epsilon)(z) f^2(C_{\varphi_r}(z)} = \overline{(1+\epsilon)(z)}\ \langle B^{z+\epsilon}_{\varphi_{r(z)}}, f^2\rangle_{A^2_{z+\epsilon.}}$

Therefore,

$$(1+\epsilon)C_{\varphi_r})^* B^{z+\epsilon}_z = \overline{(1+\epsilon)(z)}\ B^{z+\epsilon}_{\varphi_{r(z)}}.$$

**So**,

$$|(1+\epsilon)(z)|\ \left\|B^{z+\epsilon}_{\varphi_{r(z)}}\right\|_{A^{(1+\epsilon)'}_{z+\epsilon}} = \left\|\overline{(1+\epsilon)(z)} B^{z+\epsilon}_{\varphi_r}\right\|_{A^{(1+\epsilon)'}_{z+\epsilon}} = \left\|(1+\epsilon)C_{\varphi_r})^* B^{z+\epsilon}_z\right\|_{A^{(1+\epsilon)'}_{z+\epsilon}} \le \left\|(1+\epsilon)C_{\varphi_r})^*\right\|_{A^{(1+\epsilon)'}_{z+\epsilon} \to A^{(1+\epsilon)'}_{z+\epsilon}} \left\|B^{z+\epsilon}_{\varphi_r}\right\|_{A^{(1+\epsilon)'}_{z+\epsilon}}.$$

Let $M = \left\|(1+\epsilon)C_{\varphi_r})^*\right\|_{A^{(1+\epsilon)'}_{z+\epsilon} \to A^{(1+\epsilon)'}_{z+\epsilon}}$. By [16, Theorem C] , the researcher has $\left\|B^{z+\epsilon}_{\varphi_{r(z)}}\right\|_{A^{(1+\epsilon)'}_{z+\epsilon}} \le$

$$M^{(1+\epsilon)'} \left(\frac{\|B^{z+\epsilon}_z\|_{A^{(1+\epsilon)'}_{z+\epsilon}}}{\left\|B^{z+\epsilon}_{\varphi_{r(z)}}\right\|_{A^{(1+\epsilon)'}_{z+\epsilon}}}\right)^{(1+\epsilon)'} \approx M^{(1+\epsilon)'} \left(\frac{(z+\epsilon)\big(S(\varphi_r(z))\big)}{(z+\epsilon)(S(z))}\right)^{(1+\epsilon)'-1}. \qquad (17)$$

Suppose $\varphi_r = z^m \prod_{k=1}^{\epsilon} \frac{|(1+\epsilon)_k|}{(1+\epsilon)_k} \frac{(1+\epsilon)_k - z}{1-\overline{(1+\epsilon)}z}$.

**Let**

$$c = \max\{|(1+\epsilon)_k| : k = 1,2,\cdots,\epsilon\}, and\ d = \min\{|(1+\epsilon)_k| : k = 1,2,\cdots,\epsilon\}.$$

As in the proof of [2, Lemma 2.1], for $c < |z| < 1$, the researcher has

$$\frac{1-|\varphi_r(z)|^2}{1-|z|^2} \le m + 2n\frac{1+d}{1-d}.$$

By Lemma 1, there are $0 < \epsilon < \infty$ and $0 < \delta < 1$, such that $\frac{(z+\epsilon)_*(1-\epsilon)}{\epsilon^{(1+\epsilon)}}$ is increasing on $[\delta, 1[$. Let

$$r_0 = inf\{r : r > \max\{c, \delta\} \text{ and } |\varphi_r(z)| \ge \delta \text{ for all } |z| = r\}.$$

Then $r_0 = (0\ ,1)$. Obviously, the researcher has

$$\sup_{|z| \le r_0} \frac{(z+\epsilon)(S(\varphi_r)))}{(z+\epsilon)(S(z))} < \infty. \qquad (18)$$

When $|z| > r_0$, by (3), the researcher has

$$\frac{(z+\epsilon)(S(\varphi_r)))}{(z+\epsilon)(S(z))} \approx \frac{(z+\epsilon)_*(\varphi_r(z))}{(z+\epsilon)_*(z)}.$$

So, if $|\varphi_r(z)| < |z|$,

$$\frac{(z+\epsilon)_*(\varphi_r(z))}{(z+\epsilon)_*(z)} = \frac{\frac{(z+\epsilon)_*(\varphi_r(z))}{(1-|\varphi_r(z)|^{1+\epsilon}}}{\frac{(z+\epsilon)_*(z)}{(1-|z|^{1+\epsilon}}} \frac{(1-|\varphi_r(z)|^{1+\epsilon}}{(1-|z|^{1+\epsilon}} \lesssim \left(m + 2\epsilon \frac{1+d}{1-d}\right)^{1+\epsilon}. \quad (19)$$

If $|z| \leq |\varphi_r(z)| < 1$, by Lemma 1, the researcher gets

$$\frac{(z+\epsilon)_*(\varphi_r(z))}{(z+\epsilon)_*(z)} \approx \frac{(1-|\varphi_r(z)| \int_{|\varphi_{r(z)}|}^1 (z+\epsilon)(1+\epsilon)d(1+\epsilon)}{(1-|z|) \int_{|z|}^1 (z+\epsilon)(1+\epsilon)d(1+\epsilon)} \lesssim m + 2\epsilon \frac{1+d}{1-d}. \quad (20)$$

Therefore, by (17)-(20),the researcher obtains $(1+\epsilon) \in H^\infty$. The proof is complete.

To state and show the next result see[21], we introduce a set. Let $0 < \epsilon < \infty$, $(z+\epsilon) \in \mathcal{R}$ and $\varphi_r$ be an analytic self- map of $\mathbb{D}$. The researcher defines

$$X \coloneqq \{(1+\epsilon) \in H(\mathbb{D}): (1+\epsilon)C_{\varphi_r}(A^{1+\epsilon}_{z+\epsilon}) \subset A^{1+\epsilon}_{z+\epsilon}\}.$$

**Theorem 6.** Let $0 < \epsilon < \infty$ and $\varphi_r$ be an analytic self- map of $\mathbb{D}$. Suppose $(z+\epsilon) \in \mathcal{R}$ such that

(i) $\widehat{z+\epsilon}\,(\varphi_{(1+\epsilon)})(1-\epsilon))\widehat{z+\epsilon}\,(1-\epsilon) \lesssim \widehat{z+\epsilon}(1+\epsilon)$, for all $1 \leq \epsilon \leq -\epsilon < 0$, here $\varphi_{(1+\epsilon)}(1-\epsilon) = \frac{2}{\epsilon}$;

(ii) $2A + AB - B > 0$, where $A = \liminf\limits_{\epsilon \to 0} \frac{\int_{1+\epsilon}^1 (z+\epsilon)(1+\epsilon)d(1+\epsilon)}{\epsilon(z+\epsilon)(1+\epsilon)}$ and

$$B = \limsup_{\epsilon \to 0} \frac{\int_{1+\epsilon}^1 (z+\epsilon)(1+\epsilon)d(1+\epsilon)}{\epsilon(z+\epsilon)(1+\epsilon)}.$$

If $X = H^\infty$, then $\varphi_r$ is a finite Blaschke product.

Proof. By [13, proposition 18], for any $(z+\epsilon) \in \mathbb{D}$ and $\varphi_r$ is an analytic self- map of $\mathbb{D}$, $\varphi_r$: $A^{1+\epsilon}_{z+\epsilon} \to A^{1+\epsilon}_{z+\epsilon}$ is bounded. So, for any $(1+\epsilon) \in X$, the researcher defines $\|1+\epsilon\|_X = \|(1+\epsilon)(\varphi_r)\|_{A^{1+\epsilon}_{z+\epsilon} \to A^{1+\epsilon}_{z+\epsilon}}$. Let $\{(1+\epsilon)_\epsilon\}$ be a Cauchy sequence in $X$. Then $\{(1+\epsilon)_\epsilon C_{\varphi_r}\}$ is a Cauchy sequence in $B(A^{1+\epsilon}_{z+\epsilon})$. Here, $B(A^{1+\epsilon}_{z+\epsilon})$ means the set of bounded operations on $A^{1+\epsilon}_{z+\epsilon}$. So, there exists a $T \in B(A^{1+\epsilon}_{z+\epsilon})$, such that $\lim_{\epsilon \to \infty}((1+\epsilon)_\epsilon C_{\varphi_r}) = T$. Since $f^2(z) = 1 \in A^{1+\epsilon}_{z+\epsilon}$.

$(1+\epsilon) \coloneqq T(1) \in A^{1+\epsilon}_{z+\epsilon}$, and $\lim_{\epsilon \to \infty}\|(1+\epsilon)_\epsilon - (1+\epsilon)\|_{A^{1+\epsilon}_{z+\epsilon}} = 0$

Therefore, for all $f^2 \in A^{1+\epsilon}_{z+\epsilon}$,

$$\lim_{\epsilon \to \infty}(1+\epsilon)_\epsilon(z)f^2(\varphi_r)(z)) = (1+\epsilon)(z)f^2(\varphi_r)(z)).$$

Since $\lim_{\epsilon \to \infty}\|(1+\epsilon)_\epsilon \varphi_r f^2 - Tf^2\|_{A^{1+\epsilon}_{z+\epsilon}} = 0$,

$$\lim_{\epsilon \to \infty}(1+\epsilon)_\epsilon f^2 \varphi_r(z) = (T f^2)(z).$$

So, the researcher has $Tf^2 = (1+\epsilon)C_{\varphi_r}f^2$ for all $f^2 \in A^{1+\epsilon}_{z+\epsilon}$. There fore $(1+\epsilon) \in X$. That is to say, $X$ is complete under the norm $\|\cdot\|_X$.

Since $X = H^\infty$, $C_{\varphi_r} \in B(A^{1+\epsilon}_{z+\epsilon})$. Then for all $(1+\epsilon) \in X$, $\|1+\epsilon\|_X \leq \|1+\epsilon\|_{H^\infty}\|C_{\varphi_r}\|_{A^{1+\epsilon}_{z+\epsilon} \to A^{1+\epsilon}_{z+\epsilon}}$. By Inverse Mapping Theorem, $\|1+\epsilon\|_X \leq \|1+\epsilon\|_{H^\infty}$.

By $(z+\epsilon) \in \mathcal{R}$, the researcher has $AB > 0$. Therefore $\frac{2+B}{B} > \frac{1}{A}$. So, there exists $0 < \epsilon < A$ such that $\frac{2}{B+\epsilon} + 1 + \frac{1}{A-\epsilon} > 0$.

Let $(1+\epsilon) = \frac{1-(B+\epsilon)}{B+\epsilon}$ and $(1+\epsilon) = \frac{1-(A-\epsilon)}{A-\epsilon}$. Then $3+\epsilon > 0$. By the proof of [11, Lemma 1.1], there is a $0 < \delta < 1$ such that

$\frac{(z+\epsilon)(1-\epsilon)}{\epsilon^{1+\epsilon}}$ is essential decreasing on $[\delta, 1[$ and $\lim_{\epsilon \to 0}\frac{(z+\epsilon)(1-\epsilon)}{\epsilon^{1+\epsilon}} = 0$;

**and**

$\frac{(z+\epsilon)(1-\epsilon)}{\epsilon^{1+\epsilon}}$ is essential decreasing on $[\delta, 1[$ and $\lim_{\epsilon \to 0}\frac{(z+\epsilon)(1-\epsilon)}{\epsilon^{1+\epsilon}} = \infty$.

**Let**

$$\mu(z) = \begin{cases} (z+\epsilon)(z), & \delta \leq |z| < 1; \\ \dfrac{(z+\epsilon)(\delta)(1-|z|)^{1+\epsilon}}{(1-\delta)^{1+\epsilon}}, & |z| < \delta. \end{cases}$$

Then it is easy to check that the following statements hold.

(i) $\frac{\mu(1-\epsilon)}{\epsilon^{1+\epsilon}}$ is essential decreasing on $[0,1[$ and $\lim_{\epsilon\to 0}\frac{\mu(1-\epsilon)}{\epsilon^{1+\epsilon}}=0$;

(ii) $\frac{\mu(1-\epsilon)}{\epsilon^{1+\epsilon}}$ is essential decreasing on $[0,1[$ and $\lim_{\epsilon\to 0}\frac{\mu(1-\epsilon)}{\epsilon^{1+\epsilon}}=\infty$;

(iii) $\mu\in\mathcal{R}, A=\liminf_{\epsilon\to 0}\frac{\int_{1+\epsilon}^{1}\mu(1+\epsilon)d(1+\epsilon)}{\epsilon\mu(1+\epsilon)}$ and $B=\limsup_{\epsilon\to 0}\frac{\int_{1+\epsilon}^{1}\mu(1+\epsilon)d(1+\epsilon)}{\epsilon\mu(1+\epsilon)}$;

(iv) $\|f^2\|_{A^{1+\epsilon}_{\mu}}\approx\|f^2\|_{A^{1+\epsilon}_{z+\epsilon}}$ and $\hat{\mu}(\varphi_{(1+\epsilon)}(1-\epsilon))\hat{\mu}(1-\epsilon)\lesssim\hat{\mu}(1+\epsilon)$, for all $1\le\epsilon<0$.

Therefore, without loss of generality, let $\delta=0$. So, the researcher has

$$\frac{(z+\epsilon)(\varphi_{r_{(z+\epsilon)}}(z))}{(z+\epsilon)(z)}=\frac{\frac{(z+\epsilon)(\varphi_{r_{(z+\epsilon)}}(z))}{1-\left|(\varphi_{r_{(z+\epsilon)}}(z)\right|^{1+\epsilon}}}{\frac{(z+\epsilon)(z)}{1-|z|^{1+\epsilon}}}\lesssim\left(\frac{1-\left|(\varphi_{r_{(z+\epsilon)}}(z)\right|^{2}}{1-|z|}\right)^{1+\epsilon},\ \text{when } |\varphi_r(z)|>|z|,$$

**and**

$$\frac{(z+\epsilon)(\varphi_{r_{(z+\epsilon)}}(z))}{(z+\epsilon)(z)}=\frac{\frac{(z+\epsilon)(\varphi_{r_{(z+\epsilon)}}(z))}{1-\left|(\varphi_{r_{(z+\epsilon)}}(z)\right|^{\frac{1}{2}+2\epsilon}}}{\frac{(z+\epsilon)(z)}{1-|z|^{\frac{1}{2}+2\epsilon}}}\lesssim\left(\frac{1-\left|(\varphi_{r_{(z+\epsilon)}}(z)\right|^{2}}{1-|z|}\right)^{\frac{1}{2}+2\epsilon},\ \text{when } |\varphi_r(z)|\le|z|,$$

**Therefore,**

$$\frac{(z+\epsilon)(\varphi_{r_{(z+\epsilon)}}(z))}{(z+\epsilon)(z)}\lesssim\left(\frac{1-\left|(\varphi_{r_{(z+\epsilon)}}(z)\right|^{2}}{1-|z|^2}\right)^{1+\epsilon}+\left(\frac{1-\left|(\varphi_{r_{(z+\epsilon)}}(z)\right|^{2}}{1-|z|^2}\right)^{\frac{1}{2}+2\epsilon}=$$

$$\left(\frac{1-[z+\epsilon]^2}{|1-(\overline{z+\epsilon})(z)^2|}\right)^{1+\epsilon}+\left(\frac{1-[z+\epsilon]^2}{|1-(\overline{z+\epsilon})(z)^2|}\right)^{\frac{1}{2}+2\epsilon}. \qquad (21)$$

**Let** $\alpha=\frac{6+2\epsilon}{1+2\epsilon}$ and $(1+\epsilon)_{(z+\epsilon)}(z)=\left(\frac{1}{1-(\overline{z+\epsilon})(z)}\right)^{\alpha}$. Then $\|(1+\epsilon)_{z+\epsilon}\|_{\infty}^{1+\epsilon}=\frac{1}{(1-|z+\epsilon|)^{6+2\epsilon}}$. For all $f^2\in A^{1+\epsilon}_{z+\epsilon}$, by (21), the researcher has

$$\begin{aligned}\left\|(1+\epsilon)_{z+\epsilon}C_{\varphi_r}f^2\right\|^{1+\epsilon}_{A^{1+\epsilon}_{z+\epsilon}}&=\int_{\mathbb{D}}\frac{1}{\left|1-\overline{(z+\epsilon)}z\right|^{(1+\epsilon)(\alpha)}}|f^2\circ\varphi_r(z)|^{1+\epsilon}\,(z+\epsilon)(z)dA(z)\\&=\int_{\mathbb{D}}\frac{1}{\left|1-\overline{(z+\epsilon)}\varphi_{r_{(z+\epsilon)}}(z)\right|^{(1+\epsilon)\alpha}}|f^2\circ\varphi_r\circ\varphi_{z+\epsilon}(z)|^{1+\epsilon}\,|\varphi'_{z+\epsilon}(z)|^2\;\omega(\varphi_{z+\epsilon})(z)dA(z)\\&\lesssim\frac{1}{(1-|z+\epsilon|^2)^{3+\epsilon}}\int_{\mathbb{D}}|f^2\circ\varphi_r\circ\varphi_{z+\epsilon}(z)|^{1+\epsilon}\left(1+\left(\frac{1-|z+\epsilon|^2}{\left|1-\overline{(z+\epsilon)}z\right|^2}\right)^{-\frac{1}{2}+\epsilon}\right)\omega(z)dA(Z)\\&\lesssim\frac{1}{(1-|z+\epsilon|^2)^{\frac{5}{2}+\epsilon}}\int_{\mathbb{D}}|f^2\circ\varphi_r\circ\varphi_{z+\epsilon}(z)|^{1+\epsilon}\;\omega(z)dA(z)\\&\lesssim\frac{\|f^2\|^{1+\epsilon}_{A^{1+\epsilon}_{z+\epsilon}}}{(1-|z+\epsilon|^2)^{\frac{5}{2}+2\epsilon}\;(1-|\varphi_r(z|)\hat{\omega}(\varphi_r(z+\epsilon))}.\end{aligned}$$

The last inequality can be got by [13, Proposition 18]. By $|(1+\epsilon)_{z+\epsilon}|_{\infty}\;\approx\left\|(1+\epsilon)_{z+\epsilon}C_{\varphi_r}\right\|_{A^{1+\epsilon}_{z+\epsilon}\to A^{1+\epsilon}_{z+\epsilon}}$, the researcher has

$$\frac{1}{(1-|z+\epsilon|^2)^{\frac{7}{2}}}\lesssim\frac{1}{(1-|\varphi_r(z|)\hat{\omega}(\varphi_r(z+\epsilon))}.$$

By $\frac{7}{2}+\epsilon>0$, the researcher has $|\varphi_r(z+\epsilon)|\to 1$ as $|z+\epsilon|\to 1$. By Lemma 6. $\varphi_r$ is a finite Blaschke product. The proof is complete.

By Theorems 5 and 6, for some regular weight $\omega$, $X=H^{\infty}$ if and only if $\varphi_r$ is a finite Blaschke product. Here, the researcher gives two examples.

**Corollary 7.** Let $0<\epsilon<\infty$, and $\varphi_r$ be an analytic self-map of $\mathbb{D}$. Suppose $\omega$ is either $(1+\epsilon)\ or\ (\frac{1}{2}+2\epsilon)$.

$(a)\ \omega(1-\epsilon) = \epsilon^{\alpha}\left(log\frac{e}{\epsilon}\right)^{\beta}, \alpha > -1$ and $\beta \leq 0$;

$(b)\ \omega(1-\epsilon) = \exp\left(-\beta\left(log\frac{e}{\epsilon}\right)^{\alpha}\right),\ \alpha \in (0\ ,1]$ and $\beta > 0$.

Then, $= H^{\infty}$ if and only if $\varphi_r$ is a finite Blaschke product.

Proof. By [1, (4.4) and (4.5)], the weights (a) and (b) are regular.

Suppose $\varphi_r$ is a finite Blaschke product. By [13, Theorem 18 ], $C_{\varphi_r}: A^{1+\epsilon}_{z+\epsilon} \to A^{1+\epsilon}_{z+\epsilon}$ is bounded. So, $H^{\infty} \subset X$. By Theorem 5, $X \subset H^{\infty}$. Therefore, $X = H^{\infty}$.

Suppose $X = H^{\infty}$. By Bernouilli-I′ Hôpital theorem, both (a) and (b) meet the condition (ii) of Theorem 6. So, if the researcher shows that (a) and (b) meet the condition (i) of Theorem 6, then $\varphi_r$ is a finite Blaschke product.

**Condition (a).** When $1 \leq \epsilon \leq 0$ , let $\theta = \frac{1-\epsilon}{1+\epsilon}$ and

$$f^2(\theta, 1+\epsilon) = \log\frac{e\left(1-\theta(1+2\epsilon+\epsilon^2)\right)}{\epsilon\left(1+\theta(1+\epsilon)\right)}\log\frac{e}{1-\theta(1+\epsilon)}, (0 \leq \theta \leq 1, 0 < t < 1).$$

**Then**

$$f'_{\theta}(\theta, 1+\epsilon) = -\left(\frac{1+2\epsilon+\epsilon^2}{1-\theta(1+2\epsilon+\epsilon^2)} + \frac{1+\epsilon}{1+\theta(1+\epsilon)}\right)\log\frac{\mathrm{e}+1+\epsilon}{1-\theta(1+\epsilon)}\ \log\frac{e\left(1-\theta(1+2\epsilon+\epsilon^2)\right)}{\epsilon\left(1+\theta(1+\epsilon)\right)}$$

Suppose $\epsilon > -1$ , and let

$$\mathrm{g}^2(\theta, 1+\epsilon) = \frac{1-\theta(1+\epsilon)}{1+\epsilon} f'_{\theta}(\theta, 1+\epsilon) = \log\frac{e\left(1-\theta(1+2\epsilon+\epsilon^2)\right)}{\epsilon\left(1+\theta(1+\epsilon)\right)} - \left(\frac{(1+\epsilon)\left(1-\theta(1+\epsilon)\right)}{1-\theta(1+\epsilon)} + \frac{1-\theta(1+\epsilon)}{1+\theta(1+\epsilon)}\right)\log\frac{e}{1-\theta(1+\epsilon)}.$$

Then

$$f'_{\theta}(\theta, 1+\epsilon) = -\left(\frac{2\left(1+2\epsilon+\epsilon^2\right)}{1-\theta(1+2\epsilon+\epsilon^2)} + \frac{2(1+\epsilon)}{1+\theta(1+\epsilon)}\right) - \frac{(1+\epsilon)^3-(1+\epsilon)^2}{\left(1-\theta(1+2\epsilon+\epsilon^2)\right)^2} - \frac{2(1+\epsilon)}{\left(1-\theta(1+\epsilon)\right)^2}\log\frac{e}{1-\theta(1+\epsilon)} =$$

$$\frac{1}{((1+\theta(1+\epsilon))^2}\ \left(\frac{((1+\theta(1+\epsilon))^2}{((1+\theta(1+\epsilon))^2}h^2(\theta, 1+\epsilon)) + 2(1+\epsilon)k^2(\theta, 1+\epsilon)\right),$$

Where

$$h^2(\theta, 1+\epsilon) = (1+\epsilon)^2 - (1+\epsilon)^3\log\frac{e}{1-\theta(1+\epsilon)} - (2+4\epsilon+2\epsilon^2)(1-\theta(1+2\epsilon+\epsilon^2)$$

**and**

$$k^2(\theta, 1+\epsilon) = \log\frac{e}{1-\theta(1+\epsilon)} - (1+\theta(1+\epsilon).$$

**Since**

$$h'_{\theta}\ (\theta, 1+\epsilon) = \frac{(1+\epsilon)^3+(1+\epsilon)^4-2\theta\left(1+\epsilon\right)^5)}{1-\theta(1+\epsilon)} > 0, h^{2\prime}{}_{\theta}\ (\theta, 1+\epsilon) = (1+\epsilon)\left(-1+\frac{1}{1-\theta(1+\epsilon)}\right) > 0,$$

The researcher has

$$G(\theta, 1+\epsilon): (1+\theta(1+\epsilon))^2\ \ g'_{\theta}\ (\theta, 1+\epsilon)$$

Is increasing on $[0\ ,1]$ about $\theta$. Since $\lim_{\epsilon\to 0} G(I\ ,1+\epsilon)=+\infty$, there exists a $0<t<1$ such that $G(I,1+\epsilon)>0$, for all $(1+\epsilon)\in(\tau,\mathrm{I})$. If $(1+\epsilon)\in(\tau,\mathrm{I})$, by $G(0\ ,1+\epsilon)<0$, there is a $0<\nu(1+\epsilon)<1$, such that

$$G(\theta,1+\epsilon)<0, \text{ when } 0\le\theta<\nu(1+\epsilon),$$

**and**

$$G(\theta,1+\epsilon)>0, \text{ when } \nu(1+\epsilon)<\theta\le 1.$$

**Since**

$g'_\theta\ (\theta,1+\epsilon)=\frac{G(\theta,1+\epsilon)}{((1+\theta(1+\epsilon))^2}$, when $(1+\epsilon)\in(\tau,\mathrm{I})$ , $\mathrm{g}(\theta,1+\epsilon)$ is decreasing on $0,\nu(1+\epsilon))$ and increasing on $(\nu(1+\epsilon),1]$. Since

$$\mathrm{g}(0,1+\epsilon)=\log\tfrac{\mathrm{e}}{\epsilon}-(2+\epsilon)>0, \text{and } \mathrm{g}(\mathrm{I},1+\epsilon)=\tfrac{1}{2+\epsilon}\left(2+\epsilon-\log\tfrac{\mathrm{e}}{\epsilon}\right)<0.$$

So there is a $0<\mu(1+\epsilon)<1$ for every $(1+\epsilon)\in(\tau,1)$, such that, $f^2=(\theta,1+\epsilon)$ is increasing on $\left[0,\mu(1+\epsilon)\right)$ and decreasing on $\left[\mu(1+\epsilon)\right)$. Since $f^2(0,1+\epsilon)=f^2(1,1+\epsilon)$ $=\frac{e}{\epsilon}$,

$$\frac{f^2(\theta,1+\epsilon)}{\log\frac{e}{\epsilon}}\ge 1, when\ (1+\epsilon)\in(\ (\tau,1) \text{ and } 0\le\theta\le 1.$$

It is obvious that

$$\inf_{(1+\epsilon)\in[0,\tau],\theta\in[0,1]}=\frac{f^2(\theta,1+\epsilon)}{\log\frac{e}{\epsilon}}>0.$$

Therefore, the researcher has

$$C_1:=\inf_{1\le\epsilon\le 0}\frac{\log\frac{\mathrm{e}\epsilon^2}{-2\epsilon+\epsilon^2}\log\frac{\mathrm{e}}{\epsilon}}{\log\frac{\mathrm{e}}{\epsilon}}>0.$$

So, when $\alpha>-1,\beta\le 0$ and $\omega(1-\epsilon)=\epsilon^\alpha\left(\log\frac{e}{\epsilon}\right)^\beta$, the researcher has

$$\frac{\omega(\varphi_{(1+\epsilon)}(1-\epsilon))\omega(1-\epsilon)}{\omega(1+\epsilon)}\approx\left(\frac{\log\frac{\mathrm{e}\epsilon^2}{-2\epsilon+\epsilon^2}\log\frac{\mathrm{e}}{\epsilon}}{\log\frac{\mathrm{e}}{\epsilon}}\right)^\beta\le C_1^\beta.$$

Since $\omega\in\mathcal{R}$, the researcher gets

$$\frac{\widehat{\omega}(\varphi_{(1+\epsilon)}(1-\epsilon))(\widehat{\omega}(1-\epsilon)}{\widehat{\omega}(1+\epsilon)}\ \approx\ \frac{\omega(\varphi_{(1+\epsilon)}(1-\epsilon))(\omega(1-\epsilon)}{\omega(1+\epsilon)}. \qquad (22)$$

Therefore,

$$\widehat{\omega}(\varphi_{(1+\epsilon)}(1-\epsilon))\widehat{\omega}(1-\epsilon)\lesssim\widehat{\omega}(1+\epsilon), \text{ when } 1\le\epsilon\le 0.$$

**Condition(b).** Suppose $1\le\epsilon\le 0$. Since $\frac{e\epsilon^2}{2\epsilon-\epsilon^2}\ge\frac{e\epsilon}{2\epsilon-\epsilon^2}>1$, when $0<\alpha\le 1$, the researcher has

$$\log\left(\tfrac{(e\epsilon)^2}{-2\epsilon+\epsilon^2}\right)^\alpha+\log\left(\tfrac{e}{\epsilon}\right)^\alpha\ge\log\left(\tfrac{e^4\epsilon^2}{2\epsilon^2-\epsilon^3}\right)^\alpha\ge\log\left(\tfrac{e}{-\epsilon}\right)^\alpha.$$

So, when $0<\alpha\le 1,\beta>0$ and $\omega(1-\epsilon)=\exp\left(-\beta\ \left(log\frac{e}{-\epsilon}\right)^\alpha\right)$, the researcher has

$$\omega\left(\varphi_{(1+\epsilon)}(1-\epsilon)\right)\omega(1-\epsilon)\lesssim\omega(1+\epsilon).$$

By (22), the researcher gets

$$\widehat{\omega}(\varphi_{(1+\epsilon)}(1-\epsilon))\widehat{\omega}(1-\epsilon)\lesssim\widehat{\omega}(1+\epsilon), \text{ when } 1\le\epsilon\le 0.$$

The proof is complete.

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