



Sequences of Compact Differences of Composition Operators on Bergman Spaces Induced by Sequences of Doubling Weights

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Acceptance Date: 1/10/2022 Publication Date: 30/12/2022

Abstract

Bounded and compact differences of two composition operators acting from the weighted Bergman space $A_{\omega_m}^{1+2\epsilon}$ to the Lebesgue space $L_{\nu_m}^{1+\epsilon}$, where $0 \leq \epsilon < \infty$ and ω_m belongs to the class $\mathcal{D}$ of radial weights satisfying a two-sided doubling condition, are characterized. On the way to the proofs a new description of $(1 + \epsilon)$ -Carleson measures for $A_{\omega_m}^{1+2\epsilon}$, with $\epsilon > 0$ and $\omega_m \in \mathcal{D}$, involving pseudohyperbolic discs is established. This last-mentioned result generalizes the well-known characterization of $(1 + \epsilon)$ -Carleson measures for the classical weighted Bergman space $A_{\alpha}^{1+2\epsilon}$ with $0 < \epsilon < \infty$ to the setting of doubling weights. The case $\omega_m \in \widehat{\mathcal{D}}$ is also briefly discussed and an open problem concerning this case is posed.

Keywords. Bergman space, Carleson measure, doubling weight, composition operator.

Introduction and Main Results

Each analytic self-map φ_m of the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ induces the composition operator C_{φ_m} , defined by $\sum C_{\varphi_m} f_m = \sum f_m \circ \varphi_m$, acting on the space $\mathcal{H}(\mathbb{D})$ of analytic functions in $\mathbb{D}$. These operators have been extensively studied in a variety of function spaces, see for example [4, 24, 25, 29]. Some studies on topological properties of the space of composition operators attracted attention towards differences of composition operators. The question of when the difference $C_{\varphi_m} - C_{\psi_m}$ of two composition operators is compact on the Hardy space $H^{1+\epsilon}$ was posed by Shapiro and Sundberg [26]. Differences of composition operators have been studied ever since by many authors on several function spaces, see, for example, [11, 3, 6, 10]. In 2004, Nieminen and Saksman [14] showed that the compactness of $C_{\varphi_m} - C_{\psi_m}$ on the Hardy space $H^{1+\epsilon}$, with $0 \leq \epsilon < \infty$, is independent of $(1 + \epsilon)$. By using this, Choe, Choi, Koo and Yang [2] then characterized compact operators $C_{\varphi_m} - C_{\psi_m}$ on Hardy spaces by using Carleson measures in 2020. Further, Moorhouse [12] characterized the compactness of $C_{\varphi_m} - C_{\psi_m}$ on the standard weighted Bergman space $A_{\epsilon-1}^2$. Saukko [22, 23] generalized Moorhouse's results by characterizing the compactness from $A_{\epsilon-1}^{1+\epsilon}$ to $A_{2\epsilon-1}^{1+2\epsilon}$ if either $\epsilon > 0$, or $\epsilon \geq 0$. Very recently, Shi, Li and Du [28] extended Saukko's results for the complete range $0 \leq \epsilon < \infty$ and to higher dimensions.

In this paper we are interested in the compactness of $C_{\varphi_m} - C_{\psi_m}$ on weighted Bergman spaces induced by doubling weights. We proceed with necessary definitions. For $0 \leq \epsilon < \infty$ and a positive Borel measure ν_m on $\mathbb{D}$, the Lebesgue space $L_{\nu_m}^{1+\epsilon}$ consists of complex valued ν_m -measurable functions f_m on $\mathbb{D}$ such that

$$\|f_m\|_{L_{\nu_m}^{1+\epsilon}}^{1+\epsilon} = \int_{\mathbb{D}} \sum |f_m(z)|^{1+\epsilon} d\nu_m(z) < \infty.$$

If ν_m is continuous with respect to the Lebesgue measure, that is, $d\nu_m(z) = \omega_m(z) dA(z)$ for some non-negative function ω_m , then we adopt the notation $L_{\nu_m}^{1+\epsilon} = L_{\omega_m}^{1+\epsilon}$ without arising any confusion. For a nonnegative function $\omega_m \in$



$L^1([0,1))$, its extension to $\mathbb{D}$, defined by $\omega_m(z) = \omega_m(|z|)$ for all $z \in \mathbb{D}$, is called a radial weight. For $0 \leq \epsilon < \infty$ and a radial weight ω_m , the weighted Bergman space $A_{\omega_m}^{1+\epsilon}$ consists of analytic functions in $L_{\omega_m}^{1+\epsilon}$. As usual, $A_{\epsilon-1}^{1+\epsilon}$ stands for the classical weighted Bergman space induced by the standard radial weight $\omega_m(z) = (1 - |z|^2)^{\epsilon-1}$, where $0 < \epsilon < \infty$. For the theory of these spaces consult [5, 8].

For a radial weight ω_m , write $\hat{\omega}_m(z) = \int_{|z|}^1 \omega_m(s) ds$ for all $z \in \mathbb{D}$. In this paper we always assume $\hat{\omega}_m(z) > 0$, for otherwise $A_{\omega_m}^{1+\epsilon} = \mathcal{H}(\mathbb{D})$ for each $0 \leq \epsilon < \infty$. A radial weight ω_m belongs to the class $\hat{\mathcal{D}}$ if there exists a constant $C = C(\omega_m) \geq 1$ such that $\hat{\omega}_m(1 + \epsilon) \leq C \hat{\omega}_m\left(\frac{1+(1+\epsilon)}{2}\right)$ for all $-1 \leq \epsilon < 0$. Moreover, if there exist $K = K(\omega_m) > 1$ and $C = C(\omega_m) > 1$ such that $\hat{\omega}_m(1 + \epsilon) \geq C \hat{\omega}_m\left(1 - \frac{1-(1+\epsilon)}{K}\right)$ for all $-1 \leq \epsilon < 0$, then we write $\omega_m \in \tilde{\mathcal{D}}$. In other words, $\omega_m \in \tilde{\mathcal{D}}$ if there exists $K = K(\omega_m) > 1$ and $C' = C'(\omega_m) > 0$ such that

$$\hat{\omega}_m(1 + \epsilon) \leq C' \int_{1+\epsilon}^{1-\frac{1-(1+\epsilon)}{K}} \omega_m(1 + \epsilon) d(1 + \epsilon), \quad -1 \leq \epsilon < 0.$$

The intersection $\hat{\mathcal{D}} \cap \tilde{\mathcal{D}}$ is denoted by $\mathcal{D}$, and this is the class of weights that we mainly work with. For recent developments on Bergman spaces induced by weights in $\hat{\mathcal{D}}$, see [17] and the reference therein.

In this paper we consider compact differences of two composition operators from the weighted Bergman space $A_{\omega_m}^{1+\epsilon}$ to the Lebesgue space $L_{\nu_m}^{1+2\epsilon}$ when $0 \leq \epsilon < \infty$ and $\omega_m \in \hat{\mathcal{D}}$. To state the first main result, write

$$\delta_m(z) = \sum \frac{\psi_m(z) - \varphi_m(z)}{1 - \overline{\psi_m(z)}\varphi_m(z)}, \quad z \in \mathbb{D}.$$

The next result generalizes [23, Theorem 1.2] to the setting of doubling weights.

Theorem 1. Let $0 \leq \epsilon < \infty$ and $\omega_m \in \mathcal{D}$, and let ν_m be a positive Borel measure on $\mathbb{D}$. Let φ_m and ψ_m be analytic self-maps of $\mathbb{D}$. Then the following statements are equivalent:

- (i) $C_{\varphi_m} - C_{\psi_m}: A_{\omega_m}^{1+2\epsilon} \rightarrow L_{\nu_m}^{1+\epsilon}$ is bounded;
- (ii) $C_{\varphi_m} - C_{\psi_m}: A_{\omega_m}^{1+2\epsilon} \rightarrow L_{\nu_m}^{1+\epsilon}$ is compact;
- (iii) $\delta_m C_{\varphi_m}$ and $\delta_m C_{\psi_m}$ are compact (or equivalently bounded) from $A_{\omega_m}^{1+2\epsilon}$ to $L_{\nu_m}^{1+\epsilon}$.

The proof of Theorem 1 is organized as follows. We first show that $C_{\varphi_m} - C_{\psi_m}$ is compact if $\delta_m C_{\varphi_m}$ and $\delta_m C_{\psi_m}$ are bounded. The proof of this implication is straightforward and relies on the fact that the norm of $f_m \in \mathcal{H}(\mathbb{D})$ in $A_{\omega_m}^{1+2\epsilon}$ is comparable to the $L_{\omega_m}^{1+2\epsilon}$ -norm of the non-tangential maximal function $N(f_m)(z) = \sup_{\zeta \in \Gamma(z)} |f_m(\zeta)|$, where

$$\Gamma(z) = \left\{ \zeta \in \mathbb{D} : \left| \theta - \arg \zeta \right| < \frac{1}{2} \left(1 - \frac{|\zeta|}{1 + \epsilon} \right) \right\}, \quad z = (1 + \epsilon)e^{i\theta} \in \bar{\mathbb{D}} \setminus \{0\},$$

is a non-tangential approach region with vertex at z . Then, as each compact operator is bounded, the proof boils down to showing that $\delta_m C_{\varphi_m}$ and $\delta_m C_{\psi_m}$ are compact whenever $C_{\varphi_m} - C_{\psi_m}$ is bounded. This part of the proof is more laborious. We first observe that for each ρ -lattice $\{z_k\}$ the function

$$z \mapsto \sum_k \sum (a + \epsilon)_k^m \left(\frac{1 - |z_k|}{1 - \bar{z}_k z} \right)^M \frac{1}{(\omega_m(T(z_k)))^{\frac{1}{1+2\epsilon}}}$$



belongs to $A_{\omega_m}^{1+2\epsilon}$ for all $(a + \epsilon)_m = \{(a + \epsilon)_k^m\} \in \ell^{1+2\epsilon}$ and its $A_{\omega_m}^{1+2\epsilon}$ -norm is dominated by a universal constant times $\|(a + \epsilon)_m\|_{\ell^{1+2\epsilon}}$. Then we use this function for testing and apply Khinchine's inequality. After duality arguments we finally arrive to a situation where we must understand well the continuous embeddings $A_{\omega_m}^{1+2\epsilon} \subset L_{\mu_m}^{1+\epsilon}$. Recall that a positive Borel measure μ_m on $\mathbb{D}$ is a $(1 + \epsilon)$ -Carleson measure for $A_{\omega_m}^{1+2\epsilon}$ if the identity operator $I: A_{\omega_m}^{1+2\epsilon} \rightarrow L_{\mu_m}^{1+\epsilon}$ is bounded. A complete characterization of such measures in the case $\omega_m \in \widehat{\mathcal{D}}$ can be found in [18], see also [15, 16, 19]. In particular, it is known that if $\epsilon > 0$ and $\omega_m \in \widehat{\mathcal{D}}$, then μ_m is a $(1 + \epsilon)$ -Carleson measure for $A_{\omega_m}^{1+2\epsilon}$ if and only if the function

$$\zeta \mapsto \int_{\Gamma(\zeta)} \sum \frac{d\mu_m(z)}{\omega_m(T(z))} \quad (1.1)$$

belongs to $L_{\omega_m}^{\frac{1+2\epsilon}{(1+2\epsilon)-(1+\epsilon)}}$. Here and from now on $T(z) = \{\zeta \in \mathbb{D}: z \in \Gamma(\zeta)\}$ is the tent induced by $z \in \mathbb{D} \setminus \{0\}$. Further, $\omega_m(E) = \int_E \omega_m dA$ for each measurable set $E \subset \mathbb{D}$. Observe that in (1.1) we may replace the tent $T(z)$ by the Carleson square $S(z) = \{\zeta: 1 - |z| < |\zeta| < 1, |\arg \zeta - \arg z| < (1 - |z|)/2\}$ because $\omega_m(T(z)) = \omega_m(S(z))$ for all $z \in \mathbb{D} \setminus \{0\}$ if $\omega_m \in \widehat{\mathcal{D}}$. To complete the proof of Theorem 1 we will need a variant of the above characterization of Carleson measures in the case $\omega_m \in \mathcal{D}$. Our statement involves pseudohyperbolic discs and an auxiliary weight associated to ω_m . The pseudohyperbolic distance between two points a and $(a + \epsilon)_m$ in $\mathbb{D}$ is $\rho(a, (a + \epsilon)_m) = |(a - (a + \epsilon)_m)/(1 - \bar{a}(a + \epsilon)_m)|$. For $a \in \mathbb{D}$ and $-1 < \epsilon < 1$, the pseudohyperbolic disc of center a and of radius $(1 + \epsilon)$ is $\Delta(a, 1 + \epsilon) = \{z \in \mathbb{D}: \rho(a, z) < 1 + \epsilon\}$. It is well known that $\Delta(a, 1 + \epsilon)$ is an Euclidean disk centered at $(1 - (1 + \epsilon)^2)a/(1 - (1 + \epsilon)^2|a|^2)$ and of radius $(1 - |a|^2)(1 + \epsilon)/(1 - (1 + \epsilon)^2|a|^2)$. We denote $\tilde{\omega}_m(z) = \hat{\omega}_m(z)/(1 - |z|)$ for all $z \in \mathbb{D}$ and note that

$$\|f_m\|_{A_{\omega_m}^{1+2\epsilon}} = \|f_m\|_{A_{\tilde{\omega}_m}^{1+2\epsilon}}, \quad f_m \in \mathcal{H}(\mathbb{D}), \quad (1.2)$$

provided $\omega_m \in \mathcal{D}$, by [17]. Our embedding theorem generalizes the case $n = 0$ of [11, Theorem 1] to doubling weights and reads as follows.

Theorem 2. Let $0 \leq \epsilon < \infty$ and $\omega_m \in \mathcal{D}$, and let μ_m be a positive Borel measure on $\mathbb{D}$. Then the following statements are equivalent:

- (i) μ_m is a $(1 + \epsilon)$ -Carleson measure for $A_{\omega_m}^{1+2\epsilon}$;
- (ii) $I: A_{\omega_m}^{1+2\epsilon} \rightarrow L_{\mu_m}^{1+\epsilon}$ is compact;
- (iii) the function

$$\Theta_{\mu_m}^{\omega}(z) = \sum \frac{\mu_m(\Delta(z, 1 + \epsilon))}{\omega_m(S(z))}, \quad z \in \mathbb{D} \setminus \{0\},$$

belongs to $L_{\tilde{\omega}_m}^{\frac{1+2\epsilon}{(1+2\epsilon)-(1+\epsilon)}}$ for some (equivalently for all) $-1 < \epsilon < 0$.

Moreover,

$$\|I\|_{A_{\omega_m}^{1+2\epsilon} \rightarrow L_{\mu_m}^{1+\epsilon}} = \|\Theta_{\mu_m}^{\omega_m}\|_{L_{\tilde{\omega}_m}^{\frac{1+2\epsilon}{(1+2\epsilon)-(1+\epsilon)}}}. \quad (1.3)$$

We may not replace $L_{\tilde{\omega}_m}^{\frac{1+2\epsilon}{(1+2\epsilon)-(1+\epsilon)}}$ by $L_{\omega_m}^{\frac{1+2\epsilon}{(1+2\epsilon)-(1+\epsilon)}}$ in part (iii) of Theorem 2. A counter example can be constructed as follows. Write $D(z, 1 + \epsilon)$ for the Euclidean disc $\{\zeta: |\zeta - z| < 1 + \epsilon\}$. Let $(1 + \epsilon)_n = 1 - 2^{-n}$ and $A_n = D(0, (1 + \epsilon)_{n+1}) \setminus D(0, (1 + \epsilon)_n)$ for all $n \in \mathbb{N}$. Pick up an $\omega_m \in \mathcal{D}$ such that it vanishes on A_{2n} for all $n \in \mathbb{N}$. A simple example of a such weight is $\sum_{n \in \mathbb{N}} \chi_{A_{2n+1}}$. Then choose μ_m such that for some $\varepsilon > 0$ its support is contained in the union of the discs $\Delta(a_n, \varepsilon)$ which have



the property that for some fixed $-1 < \epsilon < 0$ we have $\Delta(z, 1 + \epsilon) \subset A_{2n}$ for all $z \in \Delta(a_n, \epsilon)$ and for all $n \in \mathbb{N}$. The choice $a_n = ((1 + \epsilon)_{2n} + (1 + \epsilon)_{2n+1})/2$ works if $-1 < \epsilon < 0$ and $\epsilon = \epsilon(1 + \epsilon) > 0$ are sufficiently small. Then, for such an $(1 + \epsilon)$, the norm $\|\Theta_{\mu_m}^{\omega_m}\|_{L_{\omega_m}^{\frac{1+2\epsilon}{(1+2\epsilon)-(1+\epsilon)}}}$ vanishes and thus it cannot be comparable to

$\|\Theta_{\mu_m}^{\omega_m}\|_{L_{\tilde{\omega}_m}^{\frac{1+2\epsilon}{(1+2\epsilon)-(1+\epsilon)}}}$ which is non-zero if μ_m is not a zero measure because $\tilde{\omega}_m$ is

strictly positive. Moreover, by choosing μ appropriately the norm $\|\Theta_{\mu_m}^{\omega_m}\|_{L_{\tilde{\omega}_m}^{\frac{1+2\epsilon}{(1+2\epsilon)-(1+\epsilon)}}}$ can be made infinite.

The second main result of this study concerns the case when $\epsilon > 0$ and $\omega_m \in \widehat{\mathcal{D}}$. It completes in part the main result in [10] which concerns the class $\mathcal{D}$ only. An analogue of this result was proved for the Hardy spaces in [27]. Therefore Theorem [3] takes care of the gap consisting of small Bergman spaces that exists between the Hardy and the standard weighted Bergman spaces.

Theorem 3. Let $0 \leq \epsilon < \infty$ and $\omega_m \in \widehat{\mathcal{D}}$, and let ν_m be a positive Borel measure on $\mathbb{D}$. Let φ_m and ψ_m be analytic self-maps of $\mathbb{D}$. Then $C_{\varphi_m} - C_{\psi_m}: A_{\omega_m}^{1+\epsilon} \rightarrow L_{\nu_m}^{1+2\epsilon}$ is bounded (resp. compact) if and only if δC_{φ_m} and $\delta_m C_{\psi_m}$ are bounded (resp. compact) from $A_{\omega_m}^{1+\epsilon}$ to $L_{\nu_m}^{1+2\epsilon}$.

If $\epsilon = 0$ then the boundedness (resp. compactness) of $\delta_m C_{\varphi_m}$ and $\delta_m C_{\psi_m}$ implies the same property for $C_{\varphi_m} - C_{\psi_m}$ by Proposition 4 below. Further, Proposition 5 below shows that $C_{\varphi_m} - C_{\psi_m}$ is compact if $\delta_m C_{\varphi_m}$ and $\delta_m C_{\psi_m}$ are bounded when $\epsilon > 0$. But we do not know if the boundedness of $C_{\varphi_m} - C_{\psi_m}$ implies that of $\delta_m C_{\varphi_m}$ and $\delta_m C_{\psi_m}$ if $\omega_m \in \widehat{\mathcal{D}} \setminus \mathcal{D}$ if $\epsilon \geq 0$. The methods used in this paper do not seem to give this implication and hence this case remains unsettled. The rest of the paper is organized as follows. In the next section we consider Carleson embeddings and prove Theorem 2. The sufficiency of the conditions on $\delta_m C_{\varphi_m}$ and $\delta_m C_{\psi_m}$ are established in Section 3, while Section 4 is devoted to their necessity. Finally, in Section 5 we indicate how our main findings on $C_{\varphi_m} - C_{\psi_m}$ follow from these results.

To this end, couple of words about the notation used in this paper. The letter $C = C(\cdot)$ will denote an absolute constant whose value depends on the parameters indicated in the parenthesis, and may change from one occurrence to another. We will use the notation $a \lesssim (a + \epsilon)_m$ if there exists a constant $C = C(\cdot) > 0$ such that $a \leq C(a + \epsilon)_m$, and $a \gtrsim (a + \epsilon)_m$ is understood in an analogous manner. In particular, if $a \lesssim (a + \epsilon)_m$ and $a \gtrsim (a + \epsilon)_m$, then we write $a \approx (a + \epsilon)_m$ and say that a and $(a + \epsilon)_m$ are comparable.

2. Carleson Measures

If $\omega_m \in \mathcal{D}$, then there exist constants $0 < \alpha = \alpha(\omega_m) \leq \beta = \beta(\omega_m) < \infty$ and $C = C(\omega_m) \geq 1$ such that

$$\frac{1}{C} \left(\frac{1 - (1 + \epsilon)}{1 - (1 + 2\epsilon)} \right)^\alpha \leq \frac{\widehat{\omega}_m(1 + \epsilon)}{\widehat{\omega}_m(1 + 2\epsilon)} \leq C \left(\frac{1 - (1 + \epsilon)}{1 - (1 + 2\epsilon)} \right)^\beta, \quad -1 \leq \epsilon < 0. \quad (2.1)$$

In fact, these inequalities characterize the class $\mathcal{D}$ because the right hand inequality is satisfied if and only if $\omega_m \in \widehat{\mathcal{D}}$ by [15, Lemma 2.1], while the left hand inequality describes the class $\widehat{\mathcal{D}}$ in an analogous manner [17, (2.27)]. The inequalities (2.1) will be frequently used throughout the paper.



Proof of Theorem 2. If μ_m is a $(1 + \epsilon)$ -Carleson measure for $A_{\omega_m}^{1+2\epsilon}$, then $I: A_{\omega_m}^{1+2\epsilon} \rightarrow L_{\mu_m}^{1+\epsilon}$ is automatically compact by [19, Theorem 3(iii)]. Therefore it suffices to show that (i) and (iii) are equivalent and establish (1.3).

Assume first $\Theta_{\mu_m}^{\omega_m} \in L_{\tilde{\omega}_m}^{\frac{1+2\epsilon}{(1+2\epsilon)-(1+\epsilon)}}$ for some $-1 < \epsilon < 0$. The subharmonicity of $|f_m|^{1+\epsilon}$, Fubini's theorem, Hölder's inequality and (1.2) imply

$$\begin{aligned} \|f_m\|_{L_{\mu_m}^{1+\epsilon}}^{1+\epsilon} &\lesssim \int_{\mathbb{D}} \sum \left(\int_{\Delta(z, 1+\epsilon)} \frac{|f_m(\zeta)|^{1+\epsilon}}{(1-|\zeta|^2)^2} dA(\zeta) \right) d\mu_m(z) \\ &= \int_{\mathbb{D}} \sum \left(\int_{\Delta(z, 1+\epsilon)} |f_m(\zeta)|^{1+\epsilon} \frac{\tilde{\omega}_m(\zeta)}{\omega_m(\zeta)(1-|\zeta|)} dA(\zeta) \right) d\mu_m(z) \\ &\leq \int_{\mathbb{D}} \sum |f_m(\zeta)|^{1+2\epsilon} \frac{\mu_m(\Delta(\zeta, 1+\epsilon))}{\omega_m(S(\zeta))} \tilde{\omega}_m(\zeta) dA(\zeta) \leq \sum \|f_m\|_{A_{\omega_m}^{1+2\epsilon}}^{1+\epsilon} \|\Theta_{\mu_m}^{\omega_m}\|_{L_{\tilde{\omega}_m}^{\frac{1+2\epsilon}{(1+2\epsilon)-(1+\epsilon)}}} \\ &= \sum \|f_m\|_{A_{\omega_m}^{1+2\epsilon}}^{1+2\epsilon} \|\Theta_{\mu_m}^{\omega_m}\|_{L_{\tilde{\omega}_m}^{\frac{1+2\epsilon}{(1+2\epsilon)-(1+\epsilon)}}}, \quad f_m \in A_{\omega_m}^{1+2\epsilon}. \end{aligned}$$

Therefore μ_m is a $(1 + \epsilon)$ -Carleson measure for $A_{\omega_m}^{1+2\epsilon}$ and $\|I\|_{A_{\omega_m}^{1+2\epsilon} \rightarrow L_{\mu_m}^{1+\epsilon}} \lesssim \|\Theta_{\mu_m}^{\omega_m}\|_{L_{\tilde{\omega}_m}^{\frac{1+2\epsilon}{(1+2\epsilon)-(1+\epsilon)}}}$.

Conversely, assume that μ_m is a $(1 + \epsilon)$ -Carleson measure for $A_{\omega_m}^{1+2\epsilon}$. Then (1.2) shows that μ_m is also a $(1 + \epsilon)$ -Carleson measure for $A_{\tilde{\omega}_m}^{1+2\epsilon}$ and the corresponding operator norms are comparable. Further, since $\omega_m \in \mathcal{D}$ by the hypothesis, an application of (2.1) shows that $\tilde{\omega}_m \in \mathcal{D}$. Therefore [18, Theorem 1(a)] implies

$$\|B_{\mu_m}^{\tilde{\omega}_m}\|_{L_{\tilde{\omega}_m}^{\frac{1+2\epsilon}{(1+2\epsilon)-(1+\epsilon)}}}^{\frac{1+2\epsilon}{(1+2\epsilon)-(1+\epsilon)}} = \int_{\mathbb{D}} \sum \left(\int_{\Gamma(z)} \frac{d\mu_m(\zeta)}{\tilde{\omega}_m(T(\zeta))} \right)^{\frac{1+2\epsilon}{(1+2\epsilon)-(1+\epsilon)}} \tilde{\omega}_m(z) dA(z) < \infty,$$

where

$$\Gamma(z) = \left\{ \zeta \in \mathbb{D} : |\arg \zeta - \arg z| < \frac{1}{2} \left(1 - \frac{|\zeta|}{|z|} \right) \right\}, \quad z \in \mathbb{D} \setminus \{0\},$$

is a non-tangential approach region with vertex at z . Further, by [19, Theorem 3(iii)] we have $\|I\|_{A_{\omega_m}^{1+2\epsilon} \rightarrow L_{\mu_m}^{1+\epsilon}} \gtrsim \|B_{\mu_m}^{\tilde{\omega}_m}\|_{L_{\tilde{\omega}_m}^{\frac{1+2\epsilon}{(1+2\epsilon)-(1+\epsilon)}}}$.

Let now $-1 < \epsilon < 0$ be given. For $\epsilon > 0$ and $z \in \mathbb{D} \setminus D(0, 1 - \frac{1}{K})$ write $z_K = (1 - (1 + \epsilon)(1 - |z|))e^{i \arg z}$. Pick up $(1 + \epsilon) = (1 + \epsilon)(1 + \epsilon) > 1$ and $(1 + 2\epsilon) = (1 + \epsilon)(1 + \epsilon) \in (1 - \frac{1}{K}, 1)$ sufficiently large such that $\Delta(z_K, 1 + \epsilon) \subset \Gamma(z)$ for all $z \in \mathbb{D} \setminus D(0, R)$. Straightforward applications of the left hand inequality in (2.1) show that $\tilde{\omega}_m(T(\zeta)) \lesssim \omega_m(S(\zeta))$, as $|\zeta| \rightarrow 1^-$, and $\tilde{\omega}_m(z) \lesssim \tilde{\omega}_m(z_K)$ for all $z \in \mathbb{D} \setminus D(0, R)$. In an analogous way we deduce $\omega_m(S(\zeta)) \lesssim \omega_m(S(z_K))$ for all $\zeta \in \Delta(z_K, 1 + \epsilon)$ and $z \in \mathbb{D} \setminus D(0, (1 + 2\epsilon))$ by using the right hand inequality. Therefore



$$\begin{aligned} \infty &> \|I\|_{A_{\omega_m}^{1+\epsilon} \rightarrow L_{\mu_m}^{1+\epsilon}}^{1+\epsilon} \gtrsim \int_{\mathbb{D} \setminus D(0,R)} \left(\int_{\Delta(z_K, 1+\epsilon)} \frac{d\mu_m(\zeta)}{\omega_m(T(\zeta))} \right)^{\frac{1+2\epsilon}{(1+2\epsilon)-(1+\epsilon)}} \tilde{\omega}_m(z) dA(z) \\ &\geq \int_{\mathbb{D} \setminus D(0,R)} \left(\frac{\mu_m(\Delta(z_K, 1+\epsilon))}{\omega_m(S(z_K))} \right)^{\frac{1+2\epsilon}{(1+2\epsilon)-(1+\epsilon)}} \tilde{\omega}_m(z_K) dA(z) \\ &= \int_{\mathbb{D} \setminus D(0, 1-K(1-R))} |\Theta_{\mu_m}^{\omega_m}(z)|^{\frac{1+2\epsilon}{(1+2\epsilon)-(1+\epsilon)}} \tilde{\omega}_m(z) dA(z) \end{aligned}$$

It follows that $\Theta_{\mu_m}^{\omega_m} \in L_{\tilde{\omega}_m}^{\frac{1+2\epsilon}{(1+2\epsilon)-(1+\epsilon)}}$ and $\|I\|_{A_{\omega_m}^{1+\epsilon} \rightarrow L_{\mu_m}^{1+\epsilon}}^{1+\epsilon} \gtrsim \|B_{\mu_m}^{\tilde{\omega}_m}\|_{L_{\tilde{\omega}_m}^{\frac{1+2\epsilon}{(1+2\epsilon)-(1+\epsilon)}}}^{1+\epsilon} \gtrsim \|\Theta_{\mu_m}^{\omega_m}\|_{L_{\tilde{\omega}_m}^{\frac{1+2\epsilon}{(1+2\epsilon)-(1+\epsilon)}}}^{1+\epsilon}$.

3. Sufficient Conditions

In this section we establish sufficient conditions for $C_{\varphi_m} - C_{\psi_m}: A_{\omega_m}^{1+\epsilon} \rightarrow L_{\nu_m}^{1+2\epsilon}$ to be bounded or compact. All these results are valid under the hypothesis $\omega_m \in \widehat{\mathcal{D}}$ despite the main results stated in the introduction concern only the class $\mathcal{D}$. We begin with the case $\epsilon \geq 0$.

Proposition 4. Let $0 < \epsilon < \infty$ and $\omega_m \in \widehat{\mathcal{D}}$, and let ν_m be a positive Borel measure on $\mathbb{D}$. Let φ_m and ψ_m be analytic self-maps of $\mathbb{D}$. If $\delta_m C_{\varphi_m}$ and $\delta_m C_{\psi_m}$ are bounded (resp. compact) from $A_{\omega_m}^{1+\epsilon}$ to $L_{\nu_m}^{1+2\epsilon}$, then $C_{\varphi_m} - C_{\psi_m}: A_{\omega_m}^{1+\epsilon} \rightarrow L_{\nu_m}^{1+2\epsilon}$ is bounded (resp. compact).

Proof. We begin with the statement on the boundedness. Let first $\epsilon > 0$. Let $f_m \in A_{\omega_m}^{1+\epsilon}$ with $\|f_m\|_{A_{\omega_m}^{1+\epsilon}} \leq 1$. Fix $-1 < \epsilon < 0$, and denote $E = \{z \in \mathbb{D}: |\delta_m(z)| < 1 + \epsilon\}$ and $E' = \mathbb{D} \setminus E$. Write

$$(C_{\varphi_m} - C_{\psi_m})(f_m) = (C_{\varphi_m} - C_{\psi_m})(f_m)\chi_{E'} + (C_{\varphi_m} - C_{\psi_m})(f_m)\chi_E,$$

and observe that it is enough to prove that the quantities

$$\|(C_{\varphi_m} - C_{\psi_m})(f_m)\chi_{E'}\|_{L_{\nu_m}^{1+2\epsilon}} \quad \text{and} \quad \|(C_{\varphi_m} - C_{\psi_m})(f_m)\chi_E\|_{L_{\nu_m}^{1+2\epsilon}} \quad (3.1)$$

are bounded.

We begin with considering the first quantity in (3.1). By the definition of the set E we have the estimate

$$\sum | (C_{\varphi_m} - C_{\psi_m})(f_m)\chi_{E'} | \leq \frac{1}{1+\epsilon} \sum (|\delta_m C_{\varphi_m}(f_m)| + |\delta_m C_{\psi_m}(f_m)|), \quad (3.2)$$

on $\mathbb{D}$. Since the operators $\delta_m C_{\varphi_m}$ and $\delta_m C_{\psi_m}$ both are bounded from $A_{\omega_m}^{1+\epsilon}$ to $L_{\nu_m}^{1+2\epsilon}$ by the hypothesis, the first term in (3.1) is bounded by (3.2).

We next show that also the second term in (3.1) is bounded. Let μ_m be a finite nonnegative Borel measure on $\mathbb{D}$ and h_m a measurable function on $\mathbb{D}$. For an analytic self-map φ_m of $\mathbb{D}$, the weighted pushforward measure is defined by

$$(\varphi_m)_*(h_m, \mu_m)(M) = \int_{\varphi_m^{-1}(M)} \sum h_m d\mu_m \quad (3.3)$$

for each measurable set $M \subset \mathbb{D}$. If μ_m is the Lebesgue measure, we omit the measure in the notation and write $(\varphi_m)_*(h_m)(M)$ for the left hand side of (3.3). By the measure theoretic change of variable [7, Section 39], we have $\|\delta_m C_{\varphi_m}(f_m)\|_{L_{\nu_m}^{1+2\epsilon}} =$

$\|f_m\|_{L_{(\varphi_m)_*(|\delta_m|^{1+2\epsilon}\nu_m)}^{1+2\epsilon}}$ for each $f_m \in A_{\omega_m}^{1+\epsilon}$. Therefore the identity operator from $A_{\omega_m}^{1+2\epsilon}$

to $L_{(\varphi_m)_*(|\delta_m|^{1+2\epsilon}\nu_m)}^{1+2\epsilon}$ is bounded by the hypothesis. Hence

$(\varphi_m)_*(|\delta_m|^{1+2\epsilon}\nu_m)(\Delta(\zeta, (1+2\epsilon))) \lesssim \omega_m(S(\zeta))^{\frac{1+2\epsilon}{1+\epsilon}}$ for all $\zeta \in \mathbb{D} \setminus \{0\}$ by [18,



Theorem 1(c)]. Further, by 10, Lemma 3], with $\omega_m \equiv 1$ and $\epsilon = 0$, there exists a constant $C = C((1 + \epsilon), (1 + \epsilon), (1 + 2\epsilon)) > 0$ such that

$$|f_m(z) - f_m(a)|^{1+2\epsilon} \leq C \frac{\rho(z, a)^{1+\epsilon}}{(1 - |a|)^2} \int_{\Delta(a, 1+2\epsilon)} \sum |f_m(\zeta)|^{1+\epsilon} dA(\zeta), \quad a \in \mathbb{D}, \quad z \in \Delta(a, 1 + \epsilon),$$

for all $f_m \in A_{\omega_m}^{1+\epsilon}$ with $\|f_m\|_{A_{\omega_m}^{1+\epsilon}} \leq 1$. This and Fubini's theorem yield

$$\begin{aligned} & \| (C_{\varphi_m} - C_{\psi_m})(f_m) \chi_E \|_{L_{\nu_m}^{1+2\epsilon}}^{1+2\epsilon} \\ &= \int_E \sum |f_m(\varphi_m(z)) - f_m(\psi_m(z))|^{1+2\epsilon} \nu_m(z) dA(z) \\ &\lesssim \int_E \sum \frac{|\delta_m(z)|^{1+2\epsilon}}{(1 - |\varphi_m(z)|)^2} \int_{(\varphi_m(z), 1+2\epsilon)} |f_m(\zeta)|^{1+2\epsilon} dA(\zeta) \nu_m(z) dA(z) \\ &\leq \int_{\mathbb{D}} \sum |f_m(\zeta)|^{1+2\epsilon} \left(\int_{\varphi_m^{-1}(\Delta(\zeta, 1+2\epsilon)) \cap E} \frac{|\delta_m(z)|^{1+2\epsilon}}{(1 - |\varphi_m(z)|)^2} \nu_m(z) dA(z) \right) dA(\zeta) \\ &\lesssim \int_{\mathbb{D}} \sum |f_m(\zeta)|^{1+2\epsilon} \left(\int_{\varphi_m^{-1}(\Delta(\zeta, 1+2\epsilon))} \frac{|\delta(z)|^{1+2\epsilon}}{(1 - |\zeta|)^2} \nu_m(z) dA(z) \right) dA(\zeta) \\ &= \int_{\mathbb{D}} \sum |f_m(\zeta)|^{1+2\epsilon} \frac{(\varphi_m)_*(|\delta_m|^{1+2\epsilon} \nu_m)(\Delta(\zeta, 1 + 2\epsilon))}{(1 - |\zeta|)^2} dA(\zeta) \\ &\lesssim \int_{\mathbb{D}} \sum |f_m(\zeta)|^{1+2\epsilon} \frac{\omega_m(S(\zeta))^{\frac{1+2\epsilon}{1+\epsilon}}}{(1 - |\zeta|)^2} dA(\zeta) = \int_{\mathbb{D}} \sum |f_m(\zeta)|^{1+2\epsilon} d\mu_m(\zeta). \end{aligned} \quad (3.4)$$

Standard arguments show that $\mu_m(S(a)) \lesssim \omega_m(S(a))^{\frac{1+2\epsilon}{1+\epsilon}}$ for all $a \in \mathbb{D} \setminus \{0\}$. Hence [18, Theorem 1(c)] yields $\| (C_{\varphi_m} - C_{\psi_m})(f_m) \chi_E \|_{L_{\nu_m}^{1+2\epsilon}}^{1+2\epsilon} \lesssim \sum \|f_m\|_{L_{\mu_m}^{1+2\epsilon}}^{1+2\epsilon} \lesssim \sum \|f_m\|_{A_{\omega_m}^{1+\epsilon}}^{1+2\epsilon}$. Therefore also the second term in (3.1) is bounded. This finishes the proof of the case $\epsilon > 0$.

Let now $\epsilon = 0$. By following the proof above, it suffices to show that

$$\int_S \sum \frac{(\varphi_m)_*(|\delta_m|^{1+2\epsilon} \nu_m)(\Delta(\zeta, 1 + 2\epsilon))}{(1 - |\zeta|)^2} dA(\zeta) \lesssim \sum \omega_m(S) \quad (3.5)$$

for every Carleson square $S \subset \mathbb{D}$. By the hypothesis, the identity operator from $A_{\omega_m}^{1+\epsilon}$ to $L_{(\varphi_m)_*(|\delta_m|^{1+2\epsilon} \nu_m)}^{1+\epsilon}$ is bounded, and hence $(\varphi_m)_*(|\delta_m|^{1+2\epsilon} \nu_m)(S) \lesssim \omega_m(S)$ for all S by [18, Theorem 1(b)]. But for each positive Borel measure μ_m on $\mathbb{D}$, Fubini's theorem yields

$$\begin{aligned} & \int_{S(a)} \sum \frac{\mu_m(\Delta(\zeta, 1 + 2\epsilon))}{(1 - |\zeta|)^2} dA(\zeta) \\ &= \int_{\{z \in \mathbb{D} : S(a) \cap \Delta(z, 1+2\epsilon) \neq \emptyset\}} \sum \left(\int_{S(a) \cap \Delta(z, 1+2\epsilon)} \frac{dA(\zeta)}{(1 - |\zeta|)^2} \right) d\mu_m(z) \\ &\leq \int_{S((a+\epsilon)_m)} \sum \left(\int_{\Delta(z, 1+2\epsilon)} \frac{dA(\zeta)}{(1 - |\zeta|)^2} \right) d\mu_m(z) = \mu_m(S((a+\epsilon)_m)), \quad |a| > (1 + 2\epsilon), \end{aligned} \quad (3.6)$$

where $(a + \epsilon)_m = (a + \epsilon)_m(a, 1 + 2\epsilon) \in \mathbb{D}$ satisfies $\arg(a + \epsilon)_m = \arg a$ and $1 - |(a + \epsilon)_m| = 1 - |a|$ for all $a \in \mathbb{D} \setminus \overline{D(0, 1 + 2\epsilon)}$. By applying this to $\mu_m = (\varphi_m)_*(|\delta_m|^{1+2\epsilon} \nu_m)$ and using the hypothesis $\omega_m \in \widehat{\mathcal{D}}$ we deduce



$$\int_{S(a)} \sum \frac{(\varphi_m)_*(|\delta_m|^{1+2\epsilon} v_m)(\Delta(\zeta, 1+2\epsilon))}{(1-|\zeta|)^2} dA(\zeta) \lesssim \sum (\varphi_m)_*(|\delta_m|^{1+2\epsilon} v_m)(S((a+\epsilon)_m)) \\ \lesssim \sum \omega_m(S((a+\epsilon)_m)) \lesssim \sum \omega_m(S(a)), \quad |a| > (1+2\epsilon).$$

This estimate implies (3.5), and thus the case $\epsilon = 0$ is proved.

To obtain the compactness statement, it suffices to show that the quantities

$$\|(C_{\varphi_m} - C_{\psi_m})((f_m)_n) \chi_{E'}\|_{L_{v_m}^{1+2\epsilon}}, \quad \text{and} \quad \|(C_{\varphi_m} - C_{\psi_m})((f_m)_n) \chi_E\|_{L_{v_m}^{1+2\epsilon}} \quad (3.7)$$

tend to zero as $n \rightarrow \infty$ for each sequence $\{(f_m)_n\}_{n \in \mathbb{N}}$ in $A_{\omega_m}^{1+\epsilon}$ which tends to zero uniformly on compact subsets of $\mathbb{D}$ as $n \rightarrow \infty$ and satisfies $\|(f_m)_n\|_{A_{\omega_m}^{1+\epsilon}} \leq 1$ for all $n \in \mathbb{N}$. Since $\delta_m C_{\varphi_m}$ and $\delta_m C_{\psi_m}$ are compact from $A_{\omega_m}^{1+\epsilon}$ to $L_{v_m}^{1+2\epsilon}$ by the hypothesis, an application of (3.2) to $f_m = (f_m)_n$ shows that the first quantity in (3.7) tends to zero as $n \rightarrow \infty$. As for the second quantity, observe that (3.4) implies

$$\sum \|(C_{\varphi_m} - C_{\psi_m})((f_m)_n) \chi_E\|_{L_{v_m}^{1+2\epsilon}}^{1+2\epsilon} \\ \lesssim \int_{\mathbb{D}} \sum |(f_m)_n(\zeta)|^{1+2\epsilon} \frac{(\varphi_m)_*(|\delta_m|^{1+2\epsilon} v_m)(\Delta(\zeta, 1+2\epsilon))}{(1-|\zeta|)^2} dA(\zeta), \quad n \in \mathbb{N}. \quad (3.8)$$

Let first $\epsilon > 0$. Since the identity operator from $A_{\omega_m}^{1+\epsilon}$ to $L_{(\varphi_m)_*(|\delta_m|^{1+2\epsilon} v_m)}^{1+2\epsilon}$ is compact by the hypothesis, we have $(\varphi_m)_*(|\delta_m|^{1+2\epsilon} v_m)(S(\zeta))/\omega_m(S(\zeta))^{\frac{1+2\epsilon}{1+\epsilon}} \rightarrow 0$ as $|\zeta| \rightarrow 1^-$ by [19, Theorem 3(ii)]. Now, for each $\zeta \in \mathbb{D} \setminus \{0\}$ pick up $\zeta' = \zeta'(\zeta, 1+2\epsilon) \in \mathbb{D}$ such that $\arg \zeta' = \arg \zeta$, $\Delta(\zeta', 1+2\epsilon) \subset S(\zeta)$ and $1-|\zeta'| \approx 1-|\zeta|$ for all $\zeta \in \mathbb{D} \setminus \{0\}$. Then

$$\sum \frac{(\varphi_m)_*(|\delta_m|^{1+2\epsilon} v_m)(S(\zeta))}{\omega_m(S(\zeta))^{\frac{1+2\epsilon}{1+\epsilon}}} \geq \sum \frac{(\varphi_m)_*(|\delta_m|^{1+2\epsilon} v_m)(\Delta(\zeta', 1+2\epsilon))}{\omega_m(S(\zeta'))^{\frac{1+2\epsilon}{1+\epsilon}}} \\ \geq \sum \frac{(\varphi_m)_*(|\delta_m|^{1+2\epsilon} v_m)(\Delta(\zeta', 1+2\epsilon))}{\omega_m(S(\zeta'))^{\frac{1+2\epsilon}{1+\epsilon}}}, \quad \zeta \in \mathbb{D} \setminus \{0\},$$

and hence $\sup_{\zeta \in \mathbb{D} \setminus \{0\}} (\varphi_m)_*(|\delta_m|^{1+2\epsilon} v_m)(\Delta(\zeta, 1+2\epsilon))/\omega_m(S(\zeta))^{\frac{1+2\epsilon}{1+\epsilon}} < \infty$ and, for a given $\varepsilon > 0$, there exists $\eta = \eta(\varepsilon) \in (0, 1)$ such that $\sum (\varphi_m)_*(|\delta_m|^{1+2\epsilon} v_m)(\Delta(\zeta, 1+2\epsilon))/\omega_m(S(\zeta))^{\frac{1+2\epsilon}{1+\epsilon}} < \varepsilon$ for all $\zeta \in \mathbb{D} \setminus D(0, \eta)$. Further, by the uniform convergence, there exists $N = N(\varepsilon, \eta, 1+2\epsilon) \in \mathbb{N}$ such that $|(f_m)_n|^{1+2\epsilon} \leq \varepsilon$ on $D(0, \eta)$ for all $n \geq N$. These observations together with the proof of the boundedness case and (3.8) yield

$$\sum \|(C_{\varphi_m} - C_{\psi_m})((f_m)_n) \chi_E\|_{L_{v_m}^{1+2\epsilon}}^{1+2\epsilon} \\ \lesssim \sup_{\zeta \in D(0, \eta) \setminus \{0\}} \sum \frac{(\varphi_m)_*(|\delta_m|^{1+2\epsilon} v_m)(\Delta(\zeta, 1+2\epsilon))}{\omega_m(S(\zeta))^{\frac{1+2\epsilon}{1+\epsilon}}} \int_{D(0, \eta)} |(f_m)_n(\zeta)|^{1+2\epsilon} \frac{\omega_m(S(\zeta))^{\frac{1+2\epsilon}{1+\epsilon}}}{(1-|\zeta|)^2} dA(\zeta) \\ + \sup_{\zeta \in \mathbb{D} \setminus D(0, \eta)} \sum \frac{(\varphi_m)_*(|\delta_m|^{1+2\epsilon} v_m)(\Delta(\zeta, 1+2\epsilon))}{\omega_m(S(\zeta))^{\frac{1+2\epsilon}{1+\epsilon}}} \int_{\mathbb{D} \setminus D(0, \eta)} |(f_m)_n(\zeta)|^{1+2\epsilon} \frac{\omega_m(S(\zeta))^{\frac{1+2\epsilon}{1+\epsilon}}}{(1-|\zeta|)^2} dA(\zeta) \\ \lesssim \varepsilon \int_{\mathbb{D}} \sum \frac{\omega_m(S(\zeta))^{\frac{1+2\epsilon}{1+\epsilon}}}{(1-|\zeta|)^2} dA(\zeta) + \varepsilon \sum \|(f_m)_n\|_{A_{\omega_m}^{1+\epsilon}}^{1+2\epsilon} \lesssim \varepsilon, \quad n \geq N.$$

Thus also the second quantity in (3.7) tends to zero as $n \rightarrow \infty$ in the case $\epsilon > 0$.

Finally, let $\epsilon = 0$. The compactness of the identity operator from $A_{\omega_m}^{1+\epsilon}$ to $L_{(\varphi_m)_*(|\delta_m|^{1+\epsilon} v_m)}^{1+\epsilon}$ implies $(\varphi_m)_*(|\delta_m|^{1+\epsilon} v_m)(S(\zeta))/\omega_m(S(\zeta)) \rightarrow 0$ as $|\zeta| \rightarrow 1^-$ by



[19, Theorem 3(ii)]. By following the proof above the only different step consists of making the quantity

$$J(\eta) = \int_{\mathbb{D} \setminus D(0, \eta)} \sum |(f_m)_n(\zeta)|^{1+\epsilon} \frac{(\varphi_m)_*(|\delta_m|^{1+\epsilon} \nu_m)(\Delta(\zeta, 1+2\epsilon))}{(1-|\zeta|)^2} dA(\zeta)$$

small, uniformly in n , by choosing $\eta \in (0, 1)$ sufficiently large. But an application of [18, Theorem 1(b)] together with (3.6) yields

$$J(\eta) \lesssim \| (f_m)_n \|_{A_{\omega_m}^{1+\epsilon}}^{1+\epsilon} \sup_{(a+\epsilon)_m \in \mathbb{D} \setminus \{0\}} \frac{(\varphi_m)_*(|\delta_m|^{1+\epsilon} \nu_m)(S((a+\epsilon)_m) \setminus D(0, \eta))}{\omega_m(S((a+\epsilon)_m))}.$$

Standard arguments can now be used to make the right hand side smaller than a pregiven $\varepsilon > 0$ for η sufficiently large by using $(\varphi_m)_*(|\delta_m|^{1+\epsilon} \nu_m)(S(\zeta))/\omega_m(S(\zeta)) \rightarrow 0$ as $|\zeta| \rightarrow 1^-$, see, for example, [16 pp. 26-27] for details. This completes the proof of the proposition.

The next result is a counter part of Proposition 4 in the case $\epsilon > 0$.

Proposition 5. Let $0 \leq \epsilon < \infty$ and $\omega_m \in \widehat{\mathcal{D}}$, and let ν_m be a positive Borel measure on $\mathbb{D}$. Let φ_m and ψ_m be analytic self-maps of $\mathbb{D}$. If $\delta_m C_{\varphi_m}$ and δC_{ψ_m} are bounded from $A_{\omega_m}^{1+2\epsilon}$ to $L_{\nu_m}^{1+\epsilon}$, then $C_{\varphi_m} - C_{\psi_m}: A_{\omega_m}^{1+2\epsilon} \rightarrow L_{\nu_m}^{1+\epsilon}$ is compact.

Proof. Let $\{(f_m)_n\}$ be a bounded sequence in $A_{\omega_m}^{1+2\epsilon}$ such that $(f_m)_n \rightarrow 0$ uniformly on compact subsets of $\mathbb{D}$. Since $\delta_m C_{\varphi_m}$ and $\delta_m C_{\psi_m}$ are bounded from $A_{\omega_m}^{1+2\epsilon}$ to $L_{\nu_m}^{1+\epsilon}$ by the hypothesis, they are also compact by [19, Theorem 3(iii)], and therefore

$$\lim_{n \rightarrow \infty} \sum \left(\|\delta_m (f_m)_n(\varphi_m)\|_{L_{\nu_m}^{1+\epsilon}} + \|\delta_m (f_m)_n(\psi_m)\|_{L_{\nu_m}^{1+\epsilon}} \right) = 0. \quad (3.9)$$

Let $-1 < \epsilon < 1$, and denote $E = \{z \in \mathbb{D}: |\delta(z)| < 1 + \epsilon\}$ and $E' = \mathbb{D} \setminus E$. To prove the compactness of $C_{\varphi_m} - C_{\psi_m}: A_{\omega_m}^{1+2\epsilon} \rightarrow L_{\nu_m}^{1+\epsilon}$, it suffices to show that

$$\lim_{n \rightarrow \infty} \sum \left(\|(C_{\varphi_m} - C_{\psi_m})((f_m)_n)\chi_E\|_{L_{\nu_m}^{1+\epsilon}} + \|(C_{\varphi_m} - C_{\psi_m})((f_m)_n)\chi_{E'}\|_{L_{\nu_m}^{1+\epsilon}} \right) = 0$$

since

$$\begin{aligned} \sum \|(C_{\varphi_m} - C_{\psi_m})((f_m)_n)\|_{L_{\nu_m}^{1+\epsilon}}^{1+\epsilon} &= \sum \|(C_{\varphi_m} - C_{\psi_m})((f_m)_n)\chi_E\|_{L_{\nu_m}^{1+\epsilon}}^{1+\epsilon} \\ &\quad + \sum \|(C_{\varphi_m} - C_{\psi_m})((f_m)_n)\chi_{E'}\|_{L_{\nu_m}^{1+\epsilon}}^{1+\epsilon}. \end{aligned}$$

By using (3.2) and (3.9), it is easy to show that

$$\lim_{n \rightarrow \infty} \sum \|(C_{\varphi_m} - C_{\psi_m})((f_m)_n)\chi_{E'}\|_{L_{\nu_m}^{1+\epsilon}} = 0.$$

Further, by (3.4), we have

$$\begin{aligned} \sum \|(C_{\varphi_m} - C_{\psi_m})((f_m)_n)\chi_E\|_{L_{\nu_m}^{1+\epsilon}}^{1+\epsilon} &\lesssim \int_{\mathbb{D}} \sum |(f_m)_n(\zeta)|^{1+\epsilon} \frac{(\varphi_m)_*(|\delta_m|^{1+\epsilon} \nu_m)(\Delta(\zeta, 1+2\epsilon))}{(1-|\zeta|)^2} dA(\zeta). \end{aligned}$$

Let $\varepsilon > 0$. Since the identity operator from $A_{\omega_m}^{1+2\epsilon}$ to $L_{(\varphi_m)_*(|\delta_m|^{1+\epsilon} \nu_m)}$ is bounded by the hypothesis, [19, Theorem 3(iii)] and the dominated convergence theorem imply the existence of an $(1+2\epsilon)_0 = (1+2\epsilon)_0(\varepsilon) \in (0, 1)$ such that

$$\begin{aligned} \int_{\mathbb{D}} \sum \left(\int_{\Gamma(z) \setminus \overline{D(0, (1+2\epsilon)_0)}} \frac{(\varphi_m)_*(|\delta_m|^{1+\epsilon} \nu_m)(\Delta(\zeta, 1+2\epsilon))}{\omega_m(T(\zeta))(1-|\zeta|)^2} dA(\zeta) \right)^{\frac{1+2\epsilon}{\varepsilon(1+2\epsilon)-(1+\epsilon)}} \omega_m(z) dA(z) \\ < \frac{1+2\epsilon}{\varepsilon(1+2\epsilon)-(1+\epsilon)}. \end{aligned} \quad (3.10)$$



Further, by the uniform convergence, there exists $N = N(\varepsilon) \in \mathbb{N}$ such that $|(f_m)_n(z)| < \varepsilon^{\frac{1}{1+\varepsilon}}$ for all $n \geq N$ and $z \in \overline{D(0, (1+2\varepsilon)_0)}$. Therefore, for all $n \geq N$, by Fubini's theorem, Hölder's inequality, [16. Lemma 4.4] and (3.10), we have

$$\begin{aligned} & \sum \| (C_{\varphi_m} - C_{\psi_m})((f_m)_n) \chi_E \|_{L_{\nu_m}^{1+\varepsilon}}^{1+\varepsilon} \\ & \leq \sum \left(\int_{\overline{D(0, (1+2\varepsilon)_0)}} + \int_{\mathbb{D} \setminus \overline{D(0, (1+2\varepsilon)_0)}} \right) |(f_m)_n(\zeta)|^{1+\varepsilon} \frac{(\varphi_m)_*(|\delta_m|^{1+\varepsilon} \nu_m)(\Delta(\zeta, 1+2\varepsilon))}{(1-|\zeta|)^2} dA(\zeta) \\ & \leq \varepsilon + \int_{\mathbb{D}} \sum \left(\int_{\Gamma(z) \setminus \overline{D(0, (1+2\varepsilon)_0)}} |(f_m)_n(\zeta)|^{1+\varepsilon} \frac{(\varphi_m)_*(|\delta_m|^{1+\varepsilon} \nu_m)(\Delta(\zeta, 1+2\varepsilon))}{\omega_m(T(\zeta))(1-|\zeta|)^2} dA(\zeta) \right) \omega_m(z) dA(z) \\ & \leq \varepsilon + \int_{\mathbb{D}} \sum N((f_m)_n)^{1+\varepsilon}(z) \left(\int_{\Gamma(z) \setminus \overline{D(0, (1+2\varepsilon)_0)}} \frac{(\varphi_m)_*(|\delta_m|^{1+\varepsilon} \nu_m)(\Delta(\zeta, 1+2\varepsilon))}{\omega_m(T(\zeta))(1-|\zeta|)^2} dA(\zeta) \right) \omega_m(z) dA(z) \\ & \leq \varepsilon \sum \left(1 + \|N((f_m)_n)\|_{L_{\omega_m}^{1+2\varepsilon}}^{1+\varepsilon} \right) = \varepsilon \end{aligned}$$

Therefore $\lim_{n \rightarrow \infty} \sum \| (C_{\varphi_m} - C_{\psi_m})((f_m)_n) \chi_E \|_{L_{\nu_m}^{1+\varepsilon}} = 0$, and thus $C_{\varphi_m} - C_{\psi_m}$ is compact from $A_{\omega_m}^{1+2\varepsilon}$ to $L_{\nu_m}^{1+\varepsilon}$.

4. Necessary Conditions

In this section we establish necessary conditions for $C_{\varphi_m} - C_{\psi_m}: A_{\omega_m}^{1+2\varepsilon} \rightarrow L_{\nu_m}^{1+\varepsilon}$ to be bounded or compact. In the case $\varepsilon < 0$ we work with the whole class $\widehat{\mathcal{D}}$, but the arguments employed in the case $\varepsilon \geq 0$ rely strongly on the hypothesis $\omega_m \in \mathcal{D}$. We begin with the case $\varepsilon \leq 0$.

Proposition 6. Let either $0 \leq \varepsilon < \infty$ and $\omega_m \in \widehat{\mathcal{D}}$ or $\varepsilon = 0$ and $\omega_m \in \mathcal{D}$, and let ν_m be a positive Borel measure on $\mathbb{D}$. Let φ_m and ψ_m analytic self-maps of $\mathbb{D}$. If $C_{\varphi_m} - C_{\psi_m}: A_{\omega_m}^{1+2\varepsilon} \rightarrow L_{\nu_m}^{1+\varepsilon}$ is bounded (resp. compact), then $\delta_m C_{\varphi_m}$ and $\delta_m C_{\psi_m}$ are bounded (resp. compact) from $A_{\omega_m}^{1+2\varepsilon}$ to $L_{\nu_m}^{1+\varepsilon}$.

Proof. Let first $\varepsilon > 0$ and $\omega_m \in \widehat{\mathcal{D}}$. We begin with the boundedness and show in detail that $\delta_m C_{\varphi_m}$ is bounded from $A_{\omega_m}^{1+2\varepsilon}$ to $L_{\nu_m}^{1+\varepsilon}$. For each $a \in \mathbb{D}$, consider the function

$$(f_m)_a(z) = \left(\frac{1-|a|}{1-\bar{a}z} \right)^\gamma \sum \omega_m(S(a))^{-\frac{1}{1+\varepsilon}}, \quad z \in \mathbb{D},$$

induced by ω_m and $0 \leq \varepsilon < \infty$. Then [15, Lemma 2.1] implies that for each $\gamma = \gamma(\omega_m, 1+2\varepsilon) > 0$ sufficiently large we have $\|(f_m)_a\|_{A_{\omega_m}^{1+2\varepsilon}} = 1$ for all $a \in \mathbb{D}$. Fix

such a γ . Since $C_{\varphi_m} - C_{\psi_m}$ is bounded, we have

$$\begin{aligned} 1 &= \sum \| (f_m)_a \|_{A_{\omega_m}^{1+2\varepsilon}}^{1+\varepsilon} \approx \sum \| (C_{\varphi_m} - C_{\psi_m})((f_m)_a) \|_{L_{\nu_m}^{1+\varepsilon}}^{1+\varepsilon} \\ &= \int_{\mathbb{D}} \sum \left| \left(\frac{1-|a|}{1-\bar{a}\varphi_m(z)} \right)^\gamma - \left(\frac{1-|a|}{1-\bar{a}\psi_m(z)} \right)^\gamma \right|^{1+\varepsilon} \frac{\nu_m(z)}{\omega_m(S(a))^{\frac{1+\varepsilon}{1+2\varepsilon}}} dA(z) \\ &= \int_{\mathbb{D}} \sum \left| \frac{1-|a|}{1-\bar{a}\varphi_m(z)} \right|^{\gamma(1+\varepsilon)} \left| 1 - \left(\frac{1-\bar{a}\varphi_m(z)}{1-\bar{a}\psi_m(z)} \right)^\gamma \right|^{1+\varepsilon} \frac{\nu_m(z)}{\omega_m(S(a))^{\frac{1+\varepsilon}{1+2\varepsilon}}} dA(z). \end{aligned}$$

According to [22, p. 795], for each $0 \leq \varepsilon < \infty$ and $-1 < \varepsilon < 0$ there exist a constant $C = C(\gamma, 1+\varepsilon) > 0$ such that

$$\sum \left| 1 - \left(\frac{1-\bar{a}z}{1-\bar{a}w_m} \right)^\gamma \right| \geq \sum C|a|\rho(z, w_m), \quad z \in \Delta(a, 1+\varepsilon), \quad a, w_m \in \mathbb{D}. \quad (4.1)$$

An application of this inequality gives



$$\begin{aligned} 1 &\gtrsim \int_{\varphi_m^{-1}(\Delta(a, 1+\epsilon))} \sum \frac{|a|^{1+\epsilon} |\delta_m(z)|^{1+\epsilon}}{(\omega_m(S(a)))^{\frac{1+\epsilon}{1+2\epsilon}}} \nu_m(z) dA(z) \\ &= |a|^{1+\epsilon} \sum \frac{(\varphi_m)_*(|\delta_m|^{1+\epsilon} \nu_m)(\Delta(a, 1+\epsilon))}{(\omega_m(S(a)))^{\frac{1+\epsilon}{1+2\epsilon}}}. \end{aligned}$$

It follows that $(\varphi_m)_*(|\delta_m|^{1+\epsilon} \nu_m)$ is a bounded $(1+\epsilon)$ -Carleson measure for $A_{\omega_m}^{1+2\epsilon}$ by [18, Theorem 1(c)], and hence $\delta_m C_{\varphi_m}: A_{\omega_m}^{1+2\epsilon} \rightarrow L_{\nu_m}^{1+\epsilon}$ is bounded. The same argument shows that also $\delta_m C_{\psi_m}$ is bounded.

For the compactness statement, first observe that $(f_m)_a$ tends to zero uniformly on compact subsets of $\mathbb{D}$ as $|a| \rightarrow 1^-$. Then, if $C_{\varphi_m} - C_{\psi_m}$ is compact, we have $\lim_{|a| \rightarrow 1^-} \sum \| (C_{\varphi_m} - C_{\psi_m})((f_m)_a) \|_{L_{\nu_m}^{1+\epsilon}} = 0$. By arguing as above we deduce

$$\lim_{|a| \rightarrow 1^-} \sum \frac{(\varphi_m)_*(|\delta_m|^{1+\epsilon} \nu_m)(\Delta(a, 1+\epsilon))}{(\omega_m(S(a)))^{\frac{1+\epsilon}{1+2\epsilon}}} = 0.$$

Therefore $\delta_m C_{\varphi_m}: A_{\omega_m}^{1+2\epsilon} \rightarrow L_{\nu_m}^{1+\epsilon}$ is compact by [19, Theorem 3]. The same argument shows that also $\delta_m C_{\psi_m}$ is compact.

Let now $\epsilon = 0$ and $\omega_m \in \mathcal{D}$. The statement follows from the proof above with the modification that [10, Theorem 2], valid for $\omega_m \in \mathcal{D}$, is used instead of [18, Theorem 1(c)] and [19, Theorem 3]. The only extra step is to observe that for each $\omega_m \in \mathcal{D}$ there exists $1+\epsilon = (1+\epsilon)(\omega_m) \in (0, 1)$ such that $\omega_m(S(a)) = \omega_m(\Delta(a, 1+\epsilon))$ for all $a \in \mathbb{D} \setminus \{0\}$. This follows from (2.1). With this guidance we consider the proposition proved.

The next result establishes a counter part of Proposition 6 when $\epsilon > 0$.

Proposition 7. Let $0 \leq \epsilon < \infty$ and $\omega_m \in \mathcal{D}$, and let ν_m be a positive Borel measure on $\mathbb{D}$. Let φ_m and ψ_m be analytic self-maps of $\mathbb{D}$. If $C_{\varphi_m} - C_{\psi_m}: A_{\omega_m}^{1+2\epsilon} \rightarrow L_{\nu_m}^{1+\epsilon}$ is bounded, then $\delta_m C_{\varphi_m}$ and $\delta_m C_{\psi_m}$ are both bounded from $A_{\omega_m}^{1+2\epsilon}$ to $L_{\nu_m}^{1+\epsilon}$.

Proof. Let $\{z_k\}_{k \in \mathbb{N}}$ be a ρ -lattice such that it is ordered by increasing moduli and $z_k \neq 0$ for all k . Then by [21, Theorem 1] there exist constants $M = M(1+2\epsilon, \omega_m) > 1$ and $C = C(1+2\epsilon, \omega_m) > 0$ such that the function

$$F_m(z) = \sum_k \sum (a+\epsilon)_k^m \left(\frac{1-|z_k|}{1-\bar{z}_k z} \right)^M \frac{1}{(\omega_m(T(z_k)))^{\frac{1}{1+2\epsilon}}}, \quad z \in \mathbb{D},$$

belongs to $A_{\omega_m}^{1+2\epsilon}$ and satisfies $\sum \| F_m \|_{A_{\omega_m}^{1+2\epsilon}} \leq C \sum \| (a+\epsilon)_m \|_{\ell^{1+2\epsilon}}$ for all $(a+\epsilon)_m = \{(a+\epsilon)_k^m\} \in \ell^{1+2\epsilon}$. Since $C_{\varphi_m} - C_{\psi_m}: A_{\omega_m}^{1+2\epsilon} \rightarrow L_{\nu_m}^{1+\epsilon}$ is bounded by the hypothesis, we deduce

$$\begin{aligned} &\| (a+\epsilon)_m \|_{\ell^{1+2\epsilon}}^{1+\epsilon} \\ &\gtrsim \| F_m \|_{A_{\omega_m}^{1+2\epsilon}}^{1+\epsilon} \gtrsim \int_{\mathbb{D}} \sum | (C_{\varphi_m} - C_{\psi_m}) \circ F_m(z) |^{1+\epsilon} d\nu_m(z) \\ &= \int_{\mathbb{D}} \sum \left| \sum_k (a+\epsilon)_k^m \left(\left(\frac{1-|z_k|}{1-\bar{z}_k \varphi_m(z)} \right)^M - \left(\frac{1-|z_k|}{1-\bar{z}_k \psi_m(z)} \right)^M \right) \frac{1}{(\omega_m(T(z_k)))^{\frac{1}{1+2\epsilon}}} \right|^{1+\epsilon} d\nu_m(z), \quad (a+\epsilon)_m \in \ell^{1+2\epsilon}. \end{aligned}$$

We now replace $(a+\epsilon)_k^m$ by $(a+\epsilon)_k^m (1+\epsilon)_k (1+\epsilon)$, integrate with respect to $-1 < \epsilon < 0$, and then apply Fubini's theorem and Khinchine's inequality to get



$$\| (a + \epsilon)_m \|_{\ell^{1+2\epsilon}}^{1+\epsilon} \gtrsim \int_{\mathbb{D}} \sum_k \left(\sum_k |(a + \epsilon)_k^m|^2 \left| \frac{1 - |z_k|}{1 - \bar{z}_k \varphi_m(z)} \right|^M - \left(\frac{1 - |z_k|}{1 - \bar{z}_k \psi_m(z)} \right)^M \frac{1}{(\omega_m(T(z_k)))^{\frac{2}{1+2\epsilon}}} \right)^{\frac{1+\epsilon}{2}} dv_m(z), \quad (a + \epsilon)_m \in \ell^{1+2\epsilon}.$$

By applying (4.1) and the estimate $|1 - \bar{z}_k z| = 1 - |z_k|$, valid for all $z \in \Delta(z_k, \rho)$ and $k \in \mathbb{N}$, we obtain

$$\begin{aligned} \sum \left| \left(\frac{1 - |z_k|}{1 - \bar{z}_k \varphi_m(z)} \right)^M - \left(\frac{1 - |z_k|}{1 - \bar{z}_k \psi_m(z)} \right)^M \right| &= \sum \left| \frac{1 - |z_k|}{1 - \bar{z}_k \varphi_m(z)} \right|^M \left| 1 - \left(\frac{1 - \bar{z}_k \varphi_m(z)}{1 - \bar{z}_k \psi_m(z)} \right)^M \right| \\ &\gtrsim |z_k| \sum |\delta_m(z)| \left| \frac{1 - |z_k|}{1 - \bar{z}_k \varphi_m(z)} \right|^M \chi_{\varphi_m^{-1}(\Delta(z_k, \rho))}(z) \\ &= |z_k| \sum |\delta_m(z)| \chi_{\varphi_m^{-1}(\Delta(z_k, \rho))}(z), \quad z \in \mathbb{D}, \quad k \in \mathbb{N}, \end{aligned}$$

and hence

$$\| (a + \epsilon)_m \|_{\ell^{1+2\epsilon}}^{1+\epsilon} \gtrsim |z_1|^{1+\epsilon} \int_{\mathbb{D}} \sum_k \left(\sum_k |(a + \epsilon)_k^m|^2 |\delta_m(z)|^2 \frac{1}{(\omega_m(T(z_k)))^{\frac{2}{1+2\epsilon}}} \chi_{\varphi_m^{-1}(\Delta(z_k, \rho))}(z) \right)^{\frac{1+\epsilon}{2}} dv_m(z). \quad (4.2)$$

If $\epsilon \geq 0$ then the inequality $\sum_j c_j^x \leq (\sum_j c_j)^x$, valid for all $c_j \geq 0$ and $x \geq 1$, imply

$$\| (a + \epsilon)_m \|_{\ell^{1+2\epsilon}}^{1+\epsilon} \gtrsim \int_{\mathbb{D}} \sum_k \left(\sum_k |(a + \epsilon)_k^m|^{1+\epsilon} |\delta_m(z)|^{1+\epsilon} \frac{1}{(\omega_m(T(z_k)))^{\frac{1+\epsilon}{1+2\epsilon}}} \chi_{\varphi_m^{-1}(\Delta(z_k, \rho))}(z) \right) dv_m(z). \quad (4.3)$$

To get the same estimate for $-2 < \epsilon < 0$ we apply Hölder's inequality. It together with the fact that the number of discs $\Delta(z_k, 1 + \epsilon)$ to which each $\varphi_m(z)$ may belong to is uniformly bounded yields

$$\begin{aligned} \int_{\mathbb{D}} \sum_k \left(\sum_k |(a + \epsilon)_k^m|^{1+\epsilon} |\delta_m(z)|^{1+\epsilon} \frac{1}{(\omega_m(T(z_k)))^{\frac{1+\epsilon}{1+2\epsilon}}} \chi_{\varphi_m^{-1}(\Delta(z_k, \rho))}(z) \right) dv_m(z) \\ \leq \int_{\mathbb{D}} \sum_k \left(\sum_k |(a + \epsilon)_k^m|^2 |\delta_m(z)|^2 \frac{1}{(\omega_m(T(z_k)))^{\frac{2}{1+2\epsilon}}} \chi_{\varphi_m^{-1}(\Delta(z_k, 1+\epsilon))}(z) \right)^{\frac{1+\epsilon}{2}} \\ \cdot \left(\sum_k \chi_{\varphi_m^{-1}(\Delta(z_k, \rho))}(z) \right)^{1-\frac{1+\epsilon}{2}} dv_m(z) \\ \lesssim \int_{\mathbb{D}} \sum_k \left(\sum_k |(a + \epsilon)_k^m|^2 |\delta_m(z)|^2 \frac{1}{(\omega_m(T(z_k)))^{\frac{2}{1+2\epsilon}}} \chi_{\varphi_m^{-1}(\Delta(z_k, \rho))}(z) \right)^{\frac{1+\epsilon}{2}} dv_m(z) \end{aligned}$$

Thus (4.3) holds for each $0 \leq \epsilon < \infty$. By using Fubini's theorem we now deduce

$$\begin{aligned} \| (a + \epsilon)_m \|_{\ell^{1+2\epsilon}}^{1+\epsilon} &\gtrsim \sum_k \sum |(a + \epsilon)_k^m|^{1+\epsilon} \frac{1}{(\omega_m(T(z_k)))^{\frac{1+\epsilon}{1+2\epsilon}}} \int_{\varphi_m^{-1}(\Delta(z_k, \rho))} |\delta(z)|^{1+\epsilon} dv_m(z) \\ &= \sum_k \sum |(a + \epsilon)_k^m|^{1+\epsilon} \frac{(\varphi_m)_* (|\delta_m|^{1+\epsilon} \nu_m)(\Delta(z_k, \rho))}{(\omega_m(T(z_k)))^{\frac{1+\epsilon}{1+2\epsilon}}}, \quad (a + \epsilon)_m \in \ell^{1+2\epsilon}. \end{aligned}$$

Therefore the sequence



$$\sum_{k \in \mathbb{N}} \left\{ \frac{(\varphi_m)_*(|\delta_m|^{1+\epsilon} \nu_m)(\Delta(z_k, \rho))}{(\omega_m(T(z_k)))^{\frac{1+\epsilon}{1+2\epsilon}}} \right\}$$

belongs to $\left(\ell^{\frac{1+2\epsilon}{1+\epsilon}}\right)^* \simeq \ell^{\frac{1+2\epsilon}{(1+2\epsilon)-(1+\epsilon)}}$, and consequently

$$\sum_k \sum \left(\frac{(\varphi_m)_*(|\delta_m|^{1+\epsilon} \nu_m)(\Delta(z_k, \rho))}{(\omega_m(T(z_k)))^{\frac{1+\epsilon}{1+2\epsilon}}} \right)^{\frac{1+2\epsilon}{(1+2\epsilon)-(1+\epsilon)}} < \infty.$$

Pick up an $1 + \epsilon = (1 + \epsilon)(\rho) \in (0, 1)$ such that $\Delta(z, \rho) \subset \Delta(z_k, 1 + \epsilon)$ for all $z \in \Delta(z_k, \rho)$ and $k \in \mathbb{N}$. The right hand inequality in (2.1) shows that $\hat{\omega}_m(z) \asymp \hat{\omega}_m(z_k)$ and $\omega_m(S(z)) \asymp \omega_m(S(z_k))$ for all $z \in \Delta(z_k, \rho)$ and $k \in \mathbb{N}$. Then, as $\{z_k\}_{k \in \mathbb{N}}$ is a ρ -lattice, we deduce

$$\begin{aligned} & \int_{\mathbb{D}} \sum \left(\frac{(\varphi_m)_*(|\delta_m|^{1+\epsilon} \nu_m)(\Delta(z, \rho))}{\omega_m(S(z))} \right)^{\frac{1+2\epsilon}{(1+2\epsilon)-(1+\epsilon)}} \tilde{\omega}_m(z) dA(z) \\ &= \sum_k \int_{\Delta(z_k, \rho)} \sum \left(\frac{(\varphi_m)_*(|\delta_m|^{1+\epsilon} \nu_m)(\Delta(z, \rho))}{\omega_m(S(z))} \right)^{\frac{1+2\epsilon}{(1+2\epsilon)-(1+\epsilon)}} \tilde{\omega}_m(z) dA(z) \\ &\lesssim \sum_k \sum \left(\frac{(\varphi_m)_*(|\delta_m|^{1+\epsilon} \nu_m)(\Delta(z_k, 1 + \epsilon))}{\omega_m(S(z_k))} \right)^{\frac{1+2\epsilon}{(1+2\epsilon)-(1+\epsilon)}} \tilde{\omega}_m(z_k) (1 - |z_k|)^2 \\ &\lesssim \sum_k \sum \left(\frac{(\varphi_m)_*(|\delta_m|^{1+\epsilon} \nu_m)(\Delta(z_k, 1 + \epsilon))}{(\omega_m(S(z_k)))^{\frac{1+\epsilon}{1+2\epsilon}}} \right)^{\frac{1+2\epsilon}{(1+2\epsilon)-(1+\epsilon)}} \\ &\lesssim \sum_k \sum \left(\frac{(\varphi_m)_*(|\delta_m|^{1+\epsilon} \nu_m)(\Delta(z_k, \rho))}{(\omega_m(T(z_k)))^{\frac{1+\epsilon}{1+2\epsilon}}} \right)^{\frac{1+2\epsilon}{(1+2\epsilon)-(1+\epsilon)}} < \infty. \end{aligned}$$

Therefore $(\varphi_m)_*(|\delta_m|^{1+\epsilon} \nu_m)$ is a $(1 + \epsilon)$ -Carleson measure for $A_{\omega_m}^{1+2\epsilon}$ by Theorem 2. For the same reason, $(\psi_m)_*(|\delta_m|^{1+\epsilon} \nu_m)$ is a $(1 + \epsilon)$ -Carleson measure for $A_{\omega_m}^{1+2\epsilon}$. The proof is complete.

5. Proofs of Main Theorems

Here we shortly indicate how the main results stated in the introduction easily follow from the propositions proved in the previous two sections.

Proof of Theorem 11 The theorem follows by Propositions 5 and 7. Namely, if $\delta_m C_{\varphi_m}$ and $\delta_m C_{\psi_m}$ are bounded from $A_{\omega_m}^{1+2\epsilon}$ to $L_{\nu_m}^{1+\epsilon}$, then $C_{\varphi_m} - C_{\psi_m}: A_{\omega_m}^{1+2\epsilon} \rightarrow L_{\nu_m}^{1+\epsilon}$ is compact, and thus bounded as well, by Proposition 5. Conversely, if $C_{\varphi_m} - C_{\psi_m}: A_{\omega_m}^{1+2\epsilon} \rightarrow L_{\nu_m}^{1+\epsilon}$ is bounded, then $\delta_m C_{\varphi_m}$ and $\delta_m C_{\psi_m}$ are bounded by Proposition 7.

Proof of Theorem 3 The theorem is an immediate consequence of Propositions 4 and 6.

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