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The Stability Validity on $L_{1+\epsilon}$-Curvature

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Abstract

It is known that the $L_{1+\epsilon}$-curvature of a smooth, strictly convex body in $\mathbb{R}^{2+\epsilon}$ is constant only for origin-centered balls when $1 \neq (1+\epsilon) > -(2+\epsilon)$, and only for balls when $\epsilon = 0$. If $\epsilon = 0$, then the $L_{-(2+\epsilon)}$-curvature is constant only for origin-symmetric ellipsoids. Mohammad N. Ivaki[27] prove 'local' and 'global' stability versions of these results. For $\epsilon \geq 0$, we show by following [27] the global stability result: if the $(K+\epsilon)_{1+\epsilon}$-curvature is almost a constant, then the volume symmetric difference of $\tilde{K}$ and a translate of the unit ball B is almost zero. Here $\tilde{K}$ is the dilation of K with the same volume as the unit ball. For $\epsilon \leq 1$, we show a similar result in the class of origin-symmetric bodies in the L^2-distance. In addition, for $-(2+\epsilon) < 1 + \epsilon < 0$, we show a local stability result: There is a neighborhood of the unit ball that any smooth, strictly convex body in this neighborhood with 'almost' constant $L_{1+\epsilon}$-curvature is 'almost' the unit ball. For $\epsilon = -3/2$, we show a global stability result in $\mathbb{R}^2$ and a local stability result for $\epsilon > 0$ in the Banach-Mazur distance.

Keywords: $L_{1+\epsilon}$ curvature function, $L_{1+\epsilon}$ Minkowski inequality.

Introduction

A convex body is called compact convex subset of $\mathbb{R}^{(2+\epsilon)}$ and $(2+\epsilon)$-dimensional Euclidean space, with non-empty interior. The set of convex bodies in $\mathbb{R}^{2+\epsilon}$ is denoted by $\mathcal{K}^{2+\epsilon}$ and those with the origin contained in the interior are denoted by $\mathcal{K}_0^{2+\epsilon}$. We write $\mathcal{F}^{2+\epsilon}$, and $\mathcal{F}_0^{2+\epsilon}$ for the set of smooth, strictly convex bodies in $\mathcal{K}^{2+\epsilon}$ and $\mathcal{K}_0^{2+\epsilon}$ respectively. Moreover, B and $S^{(1+\epsilon)}$ denote the unit ball and the unit sphere of $\mathbb{R}^{2+\epsilon}$ respectively. $\tilde{K}$ denotes the dilation of K whose $(2+\epsilon)$-dimensional Lebesgue measure, $V(\tilde{K})$, equals to that of the unit ball; $V(\tilde{K}) = V(B) = \kappa_{2+\epsilon}$.

The support of the series functions of a convex body K is defined by

$$\sum h_K^m(u_m) := \max_{x\in K} \sum x \cdot u_m, \quad \forall u_m \in S^{(1+\epsilon)}.$$

For $K \in \mathcal{F}_0^{2+\epsilon}$ and $(\nu_m)_K : \partial K \to S^{(1+\epsilon)}$ be the Gauss map which takes x on the boundary of K to its unique outer unit normal vector, and let

$$(\nu_m)_K^{-1} : S^{(1+\epsilon)} \to \mathbb{R}^{2+\epsilon}$$

be the Gauss parameterization of ∂K. In this case, we have

$$\sum h_K(u_m) = \sum u_m \cdot (\nu_m)_K^{-1}(u_m).$$

We write g, ∇ for the standard round metric and the corresponding Levi-Civita connection of the unit sphere. The Gauss curvature of $\partial K, \mathcal{K}_K$, and the curvature function of $\partial K, f_K^m$ (as a functions on the unit sphere), are related to the support functions of the convex body by.

$$f_K^m = \sum \frac{1}{\mathcal{K}_K \circ (\nu_m)_K^{-1}} = \sum \frac{\det\left(\nabla_{i,j}^2 h_K^m + g_{ij} h_K^m\right)}{\det(g_{ij})}$$

The function $h_K^{(-\epsilon)m} f_K^m$ is called the $(K+\epsilon)_{1+\epsilon}$-curvature function of K.

For $K \in \mathcal{F}_0^{2+\epsilon}$ we define the scale invariant quantity

$$\mathcal{R}_{1+\epsilon}(K) = \max_{S^{(1+\epsilon)}} \left(h_K^{(-\epsilon)m} f_K^m\right) / \min_{S^{(1+\epsilon)}} \left(h_K^{((-\epsilon)m)} f_K^m\right).$$

We prove stability versions of the following theorem, which is due to a collective work of Firey, Lutwak, Andrews, Brendle, Choi, and Daskalopoulos[12,19,2,4] :

Theorem. Let $0 > \epsilon > \infty, \epsilon \neq 2+\epsilon$. If $K \in \mathcal{F}_0^{2+\epsilon}$ satisfies

$$h_K^{(-\epsilon)m} f_K^m \equiv 1,$$

then$K$ is the unit ball.

Question 1. Is there an increasing function f^m with $\lim_{\varepsilon\to 0}\sum f^m(\varepsilon) = 0$ with the following property? If $K \in \mathcal{F}_0^{2+\epsilon}$satisfies $\mathcal{R}_{1+\epsilon}(K) \leq 1+\varepsilon$, then K is $f^m(\varepsilon)$-close to a ball in a suitable sense?

For $\epsilon \geq 0$ due to the $(K+\epsilon)_{1+\epsilon}$-Minkowski inequality and a refinement of Urysohn's inequality from [23] we can obtain a global stability result in a very strong sense. The relative asymmetry of two convex bodies $K, K+\epsilon$ is defined as

$$\mathcal{A}(K, K+\epsilon) := \inf_{x\in\mathbb{R}^{2+\epsilon}} \frac{V(K\Delta(\lambda(K+\epsilon)+x))}{V(K)}, \quad \text{where } \lambda^{2+\epsilon} = \frac{V(K)}{V(K+\epsilon)}$$

And $K\Delta(K+\epsilon) = (K \setminus (K+\epsilon)) \cup ((K+\epsilon) \setminus K)$.

Theorem 1.1. Let $\epsilon \geq 0$. There exists a constant C independent of dimension with the following property. Any $K \in \mathcal{F}_0^{2+\epsilon}$ satisfies

$$\mathcal{A}(\tilde{K}, B) \leq C(2+\epsilon)^{2.5}\left(\mathcal{R}_{1+\epsilon}(K)^{\frac{1}{1+\epsilon}} - 1\right)^{\frac{1}{2}}$$

The $(K+\epsilon)_{(2+\epsilon)}$-Minkowski inequality also allows us to prove the global stability for $0 \leq \epsilon \leq 1$ in the class of origin-symmetric bodies in the $(K+\epsilon)^2$-distance. The L^2-distance of $K, K+\epsilon$ is defined by

$$\delta_2(K, K+\epsilon) = \left(\frac{1}{\omega_{2+\epsilon}}\int \sum |h_K^m - h_{K+\epsilon}^m|^2 d\sigma\right)^{\frac{1}{2}}$$

Here σ is the spherical Lebesgue measure on $S^{(1+\epsilon)}$, and ω_i is the surface area of the i-dimensional ball.

Theorem 1.2. Let $0 \leq \epsilon \leq 1$ and $K \in \mathcal{F}^{2+\epsilon}$ be origin-symmetric. There exists an origin-centered ball $B_{1+\epsilon}$ with radius $1 \leq 1+\epsilon \leq \mathcal{R}_{1+\epsilon}(K)$, such that

$$\delta_2(\tilde{K}, B_{1+\epsilon}) \leq D(\tilde{K})(1 - \mathcal{R}_{1+\epsilon}(K)^{-1})^{\frac{1}{2}}$$

Here the diameter of $\tilde{K}, D(\tilde{K})$, satisfies the inequality

$$D(\tilde{K}) \leq 2\left(\left(1 + \left(\frac{4\omega_{(1+\epsilon)}}{\omega_{2+\epsilon}}\right)^{\frac{1}{2}}\right)\mathcal{R}_{1+\epsilon}(K)\right)^3$$

For $1+\epsilon \in (-(2+\epsilon), 0)$, we also establish a local stability result. The points $e_{1+\epsilon}$ will be defined in Definition 2.1.

Theorem 1.3. Let $1+\epsilon \in (-(2+\epsilon), 0)$. There exist positive constants γ, δ, depending only on $(2+\epsilon), (1+\epsilon)$ with the following property. If $K \in \mathcal{F}_0^{2+\epsilon}$ with $e_{1+\epsilon}(K) = 0$ satisfies $\sum |h_{\lambda K}^m - 1|_{C^3} \leq \delta$ for some $\lambda > 0$, then $\delta_2(\tilde{K}, B) \leq \gamma(\mathcal{R}_{1+\epsilon}(K) - 1)$.

Remark 1.4. For the case $\epsilon = 0$, some progress recently has been made on the stability of the cone-volume measure by Böröczky and De in [7] based on the logarithmic Minkowski inequality in the class of convex bodies with many symmetries proved by Böröczky and Kalantzopoulos in [6]. Although our proof of Theorem 1.2 is independent of the existence of $(K+\epsilon)_{1+\epsilon}$-Minkowski inequality for $0 \leq \epsilon < 1$, it is worth pointing out that such an inequality exists in some particular cases: $0 < \epsilon \leq 1$ and in the class of origin-symmetric convex bodies in the plane, or in any dimension and in the class of origin-symmetric bodies for $0 < \epsilon < 1$ where $\epsilon > 0$ is some constant depending on $(2+\epsilon)$; see [9,18,21,22].

Let $K\in\mathcal{F}_0^{2+\epsilon}$. The centro-affine curvature of K, H_K, is defined by

$$H_K:=\left(h_K^{(1+\epsilon)m}f_K^m\right)^{-1}$$

It is known that $H_K(u_m)$ is (up to a constant) just a power of the volume of the origin-centered ellipsoid touching K at $(\nu_m)_K^{-1}(u_m)$ of second-order (osculating ellipsoid), and thus is an$S(K+\epsilon)(2+\epsilon)$ covariant notion. In particular, one of the key properties of the centro-affine curvature is that $\min H_K$ and $\max H_K$ are invariant under special linear transformation $S(K+\epsilon)(2+\epsilon)$. That is,

$$\min_{S^{1+\epsilon}}H_K=\min_{S^{1+\epsilon}}H_{\ell K},\max_{S^{1+\epsilon}}H_K=\max_{S^{1+\epsilon}}H_{\ell K},\quad\forall\ell\in S(K+\epsilon)(2+\epsilon).\quad(1.1)$$

A remarkable theorem of Pogorelov states that if the centro-affine curvature of a smooth, strictly convex body is constant, then the body is an origin-centered ellipsoid; cf. [13, Thm. 10.5.1], [8,10,20]. We find a stability version of this statement, for example, in the Banach-Mazur distance $d_{\mathcal{BM}}$. For two convex bodies $K, K+\epsilon$ is defined by

$$d_{\mathcal{BM}}(K,K+\epsilon)=\min\{\lambda\geq1:(K-x)\subseteq\ell((K+\epsilon)-y)\subseteq\lambda(K-x),\\ \ell\in G(K+\epsilon)(2+\epsilon),x,y\in\mathbb{R}^{2+\epsilon}\}$$

Question 2. Is there an increasing function f^m with $\lim_{\varepsilon\to0}\sum f^m(\varepsilon)=0$ with the following property? If $K\in\mathcal{F}_0^{2+\epsilon}$ satisfies

$$\mathcal{R}_{-(2+\epsilon)}(K)=\frac{\max H_K}{\min H_K}\leq1+\varepsilon$$

then$K$ is $f^m(\varepsilon)$-close to an ellipsoid in the Banach-Mazur distance.

The following theorem gives a positive answer to this question in the plane under no additional assumption.

Theorem 1.5. There exist $\gamma,\delta>0$ with the following property. If $K\in\mathcal{F}_0^2$satisfies $\mathcal{R}_{-2}(K)\leq1+\delta$, then we have

$$(d_{\mathcal{BM}}(K,B)-1)^4\leq\gamma(\mathcal{R}_{-2}(K)-1)$$

If K has its Santaló point at the origin, then

$$(d_{\mathcal{BM}}(K,B)-1)^4\leq\gamma\left(\sqrt{\mathcal{R}_{-2}(K)}-1\right)$$

Moreover, if K is origin-symmetric, then

$$d_{\mathcal{BM}}(K,B)\leq\sqrt{\mathcal{R}_{-2}(K)}$$

In this case, we may allow $\delta=\infty$.

In higher dimensions, we have the following 'local' stability result.

Theorem 1.6. There exist positive numbers γ,δ, depending only on $(2+\epsilon)$ with the following property. Suppose $K\in\mathcal{F}_0^{2+\epsilon}$ has its Santaló point at the origin, and for some $\ell\in G(K+\epsilon)(2+\epsilon)$ we have $\sum|h_{\ell K}^m-1|_{C^3}\leq\delta$. Then

$$d_{\mathcal{BM}}(K,B)\leq\gamma\left(\mathcal{R}_{-(2+\epsilon)}(K)-1\right)^{\frac{1}{3(3+\epsilon)}}+1$$

2. Background

A convex body K is said to be of class C_+^2, if its boundary hypersurface is two-times continuously differentiable and the support function is differentiable.

Let $K, K+\epsilon$ be two convex bodies with the origin of $\mathbb{R}^{2+\epsilon}$ in their interiors. We put $(1+\epsilon)\cdot K:=(1+\epsilon)^{\frac{1}{1+\epsilon}}K$ and $(1+2\epsilon)\cdot(K+\epsilon):=(1+2\epsilon)^{\frac{1}{1+\epsilon}}(K+\epsilon)$where $\epsilon>0$. For $\epsilon\geq0$, the $(K+\epsilon)_{1+\epsilon}$-linear combination $(1+\epsilon)\cdot K+_{1+\epsilon}(1+2\epsilon)\cdot(K+\epsilon)$ is defined as the convex body whose support function is given by $\left((1+\epsilon)h_K^{(1+\epsilon)m}+(1+2\epsilon)h_{K+\epsilon}^{(1+\epsilon)m}\right)^{\frac{1}{1+\epsilon}}$.

For $K,K+\epsilon\in\mathcal{K}_0^{2+\epsilon}$, the mixed $(K+\epsilon)_{1+\epsilon}$-volume $V_{1+\epsilon}(K,K+\epsilon)$ is defined as the first variation of the usual volume with respect to the $(K+\epsilon)_{1+\epsilon}$-sum:

$$\frac{2+\epsilon}{1+\epsilon}V_{1+\epsilon}(K,K+\epsilon)=\lim_{\varepsilon\to0^+}\frac{V\left(K+\ {}_{1+\epsilon}\varepsilon\cdot(K+\epsilon)\right)-V(K)}{\varepsilon}.$$

Aleksandrov, Fenchel and Jessen for $\epsilon=0$ and Lutwak [19] for $\epsilon>0$ have shown that there exists a unique Borel measure $S_{1+\epsilon}(K,\cdot)$ on $S^{1+\epsilon}, L_{1+\epsilon}$-surface area measure, such that

$$V_{1+\epsilon}(K,K+\epsilon)=\frac{1}{2+\epsilon}\int\sum h_{K+\epsilon}^{(1+\epsilon)m}(u_m)dS_{1+\epsilon}(K,u_m).$$

Moreover, $S_{1+\epsilon}(K,\cdot)$ is absolutely continuous with respect to the surface area measure of $K, S(K,\cdot)$, and has the Radon-Nikodym derivative

$$\frac{dS_{1+\epsilon}(K,\cdot)}{dS(K,\cdot)}=\sum h_K^{(-\epsilon)m}(\cdot)$$

The measure $dS_{1+\epsilon,K}=h_K^{(-\epsilon)m}dS_K$is known as the $L_{1+\epsilon}$-surface area measure. If theboundary of $K$is C_+^2, then

$$\frac{dS_K}{d\sigma}=\frac{1}{\mathcal{K}_K\circ\nu_K^{-1}}=f_K^m.$$

For $\epsilon>0$, the $L_{1+\epsilon}$-Minkowski inequality states that for convex bodies $K, K+\epsilon$ with the origin in their interiors we have

$$\frac{1}{2+\epsilon}\int\sum h_{K+\epsilon}^{(1+\epsilon)m}dS_{1+\epsilon}(K)\geq V(K)^{\frac{1+\epsilon}{2+\epsilon}}V(K+\epsilon)^{\frac{1+\epsilon}{2+\epsilon}},$$

with equality holds if and only if K and $K+\epsilon$ are dilates (i.e. for some $\lambda>0, K=\lambda(K+\epsilon)$; see [19]. For $\epsilon=0$, the same inequality holds for all $K, K+\epsilon\in\mathcal{K}^{2+\epsilon}$, and equality holds if and only if K is homothetic to $(K+\epsilon)$.

The polar body, K^*, of $K\in\mathcal{K}_0^{2+\epsilon}$ is the convex body defined by

$$K^*=\{y\in\mathbb{R}^{2+\epsilon}: x\cdot y\leq1,\forall x\in K\}$$

All geometric quantities associated with the polar body are furnished by $*$. For $x\in$int K, let $K^x:=(K-x)^*$. The Santaló point of K, denoted by $s=s(K)$, is the unique point in int K such that

$$V(K^s)\leq V(K^x),\quad\forall x\in\operatorname{int}K$$

If $K=-K$, then $s(K)=0$ and $K^*=K^s$.

The Blaschke-Santaló inequality states that

$$V(K^s)V(K)\leq V(B)^2$$

and equality holds if and only if K is an ellipsoid.

Definition 2.1. The $(K+\epsilon)_{1+\epsilon}$-widths of $K\in\mathcal{K}^n$are defined as follows.

(1) For $\epsilon>0$: $\mathcal{E}_{1+\epsilon}(K)=\frac{1}{\omega_{2+\epsilon}}\inf_{x\in\operatorname{int}K}\int h_{K-x}^{1+\epsilon}d\sigma$.

(2) For $\epsilon=-1$: $\mathcal{E}_0(K)=\frac{1}{\omega_{2+\epsilon}}\sup_{x\in\operatorname{int}K}\int\sum\log h_{K-x}^m d\sigma$.

(3) For $0<\epsilon<1$: $\mathcal{E}_{1+\epsilon}(K)=\frac{1}{\omega_{2+\epsilon}}\sup_{x\in\operatorname{int}K}\int h_{K-x}^{(1+\epsilon)m}d\sigma$.

(4) For $0\leq\epsilon<(2+\epsilon)$: $\mathcal{E}_{1+\epsilon}(K)=\frac{1}{\omega_{2+\epsilon}}\inf_{x\in\operatorname{int}K}\int\sum h_{K-x}^{(1+\epsilon)m}d\sigma$.

Here $\omega_{2+\epsilon}=(2+\epsilon)\kappa_{2+\epsilon}=\int d\sigma$

Here, $e_{1+\epsilon}$ denotes the unique point at which the corresponding sup or inf is attained. The points $e_{1+\epsilon}$ are always in the interior of the convex body; see e.g. [16, Lem. 3.1]. If K is origin-symmetric, then $e_{1+\epsilon}(K)$ lies at the origin.

For $\epsilon\geq0$ by the $L_{1+\epsilon}$-Minkowski inequality we have

$$\mathcal{E}_{1+\epsilon}(\tilde{K})\geq1.\tag{2.1}$$

For $0<\epsilon\leq2+\epsilon$ by the Blaschke-Santaló inequality,

$$\mathcal{E}_0(\tilde{K})\geq0,\quad\mathcal{E}_{1+\epsilon}(\tilde{K})\leq1,\tag{2.2}$$

and equality holds when K is a ball. Moreover, for $\epsilon < 1$ we have

$$\begin{aligned}\mathcal{E}_{1+\epsilon}(\tilde{K})\mathcal{E}_{-1+\epsilon}(\tilde{K}) &= \frac{1}{\omega_{2+\epsilon}^2}\int \sum h_{\tilde{K}-e_{1+\epsilon}(\tilde{K})}^{(1+\epsilon)m} d\sigma \int h_{\tilde{K}-e_{-1+\epsilon}(\tilde{K})}^{-(1+\epsilon)m} d\sigma \qquad (2.3)\\ &\geq \frac{1}{\omega_{2+\epsilon}^2}\int \sum h_{\tilde{K}-e_{-1+\epsilon}(\tilde{K})}^{(1+\epsilon)m} d\sigma \int h_{\tilde{K}-e_{-1+\epsilon}(\tilde{K})}^{-(1+\epsilon)m} d\sigma \geq 1,\end{aligned}$$

where we used the definition of $e_{1+\epsilon}$ in the last line. Therefore we obtain

$$\mathcal{E}_{1+\epsilon}(\tilde{K}) \geq 1, \qquad (2.4)$$

and the equality holds only for balls.

We conclude by remarking that $\mathcal{E}_{1+\epsilon}$ enjoys the second Eojasiewicz-Simon gradient inequality; see [25,26].

6. Stability of the width functionals

We show the stability of the inequalities (2.1) and (2.2)($\epsilon \neq -1$) (see [27]).

Lemma 3.1. Suppose $1+\epsilon \in [-(2+\epsilon), 0)$. Let $K \in \mathcal{K}^{2+\epsilon}$with $V(K) = V(B)$. Then

$$|e_{1+\epsilon}(K) - s(K)|^2 \leq c_0(1-\mathcal{E}_{1+\epsilon}(K))D(K)^{1+\epsilon}$$

where$c_0^{-1} := \frac{(1+\epsilon)(\epsilon)}{2\omega_{2+\epsilon}}\int (u_m \cdot v_m)^2 d\sigma(u_m) = \frac{(1+\epsilon)(\epsilon)}{2(2+\epsilon)}$ for any vector v_m, and $D(K)$ denotes the diameter of K.

Proof. We may suppose $e_{1+\epsilon}(K) \neq s(K)$. Define $v_m = -\frac{e_{1+\epsilon}(K)-s(K)}{|e_{1+\epsilon}(K)-s(K)|}$ and

$$e(t) = e_{1+\epsilon}(K) + tv_m, \quad t \in [0, |e_{1+\epsilon}(K) - s(K)|].$$

Let us denote the support function of $K - e(t)$ by h_t^m and

$$E(t) := \frac{1}{\omega_{2+\epsilon}}\int \sum h_t^{(1+\epsilon)m} d\sigma.$$

Note that $E(0) = \mathcal{E}_{1+\epsilon}(K), E'(0) = 0$ and the second derivative of E is given by

$$E''(t) = \frac{(1+\epsilon)(\epsilon)}{\omega_{2+\epsilon}}\int \sum h_t^{(\epsilon-1)m}(u_m)(u_m \cdot v_m)^2 d\sigma(u_m).$$

Due to $h_t^m \leq D(K)$ we obtain

$$D(K)^{\epsilon-1}|e_{1+\epsilon}(K) - s(K)|^2 \leq c_0\left(\frac{1}{\omega_{2+\epsilon}}\int \sum h_{K-s(K)}^{(1+\epsilon)m} d\sigma - \mathcal{E}_{1+\epsilon}(K)\right)$$

Now the claim follows from the Blaschke-Santaló inequality. We have the following (see [27])

Theorem 3.2. The following statements hold.

(1) Let $\epsilon \geq 0$. If $\mathcal{E}_{1+\epsilon}(\tilde{K}) \leq 1+\varepsilon$, then

$$\mathcal{A}(\tilde{K}, B)^2 \leq C(2+\epsilon)^5\left((1+\varepsilon)^{\frac{1}{1+\epsilon}} - 1\right).$$

Here C is a universal constant that does not depend on $(2+\epsilon)$.

(2) Let $1+\epsilon \in (-(2+\epsilon), 0)$. If $\mathcal{E}_{1+\epsilon}(\tilde{K}) \geq 1-\varepsilon$, then there exists an origin-centered ball of radius $(1+\epsilon), B_{1+\epsilon}$, such that

$$\delta_2(\tilde{K} - e_{1+\epsilon}(\tilde{K}), B_{1+\epsilon}) \leq \left(2c_1(D(\tilde{K}) + (1+\epsilon))^{(3+\epsilon)}\varepsilon\right)^{\frac{1}{2}} + \left(c_0 D(\tilde{K})^{1-\epsilon}\varepsilon\right)^{\frac{1}{2}}$$

Moreover, if $\tilde{K}$ is origin-symmetric, then the last term on the right-hand-side can be dropped and $D(\tilde{K})$ can be replaced by $\frac{1}{2}D(\tilde{K})$. Here

$$1 \leq (1+\epsilon) \leq (1-\varepsilon)^{\frac{1}{1+\epsilon}}, \quad c_1 := \max\left\{\frac{2+\epsilon}{(3+2\epsilon)}, -\frac{2+\epsilon}{1+\epsilon}\right\}$$

andc_0 is the constant from Lemma 3.1.

Proof.*Case* $\epsilon \geq 0$: Since $\mathcal{E}_{1+\epsilon}(\tilde{K}) \leq 1+\varepsilon$, we have

$$\frac{1}{\omega_{2+\epsilon}}\int \sum h_{\tilde{K}}^m d\sigma \leq \mathcal{E}_{1+\epsilon}(\tilde{K})^{\frac{1}{1+\epsilon}} \leq (1+\varepsilon)^{\frac{1}{1+\epsilon}}$$

The refinement of Urysohn's inequality in [23] completes the proof.

Case $-(2+\epsilon)<1+\epsilon<0$: Assume $V(K)=V(B)$. Denote the support function of $K-e_{1+\epsilon}(K)$ by $h_{1+\epsilon}^m$ and the support function of $K-s(K)$by h_s^m. Since $s(K), e_{1+\epsilon}(K)$ are in the interior of K, both h_s^m and $h_{1+\epsilon}^m$ are positive functions.
Let us put

$$f^m=h_s^{(1+\epsilon)m}, g=1,(1+\epsilon)^2=-(2+\epsilon),(1+2\epsilon)=\frac{2+\epsilon}{(3+2\epsilon)}, c_1 = \max\{1+\epsilon, 1+2\epsilon\}$$

By [1, Thm. 2.2], we have

$$\sum\frac{\int \quad h_s^{(1+\epsilon)m}d\sigma}{\left(\int \quad \frac{1}{h_s^{(2+\epsilon)m}}d\sigma\right)^{-\frac{1+\epsilon}{2+\epsilon}}\omega_{2+\epsilon}^{\frac{3+2\epsilon}{2+\epsilon}}}\le 1-\frac{1}{c_1}\sum\left|\frac{h_s^{-\left(\frac{2+\epsilon}{2}\right)m}}{\left(\int \quad \frac{1}{h_s^{(2+\epsilon)m}}d\sigma\right)^{\frac{1}{2}}}-\frac{1}{\omega_{2+\epsilon}^{\frac{1}{2}}}\right|_{(K+\epsilon)^2}^2 . \quad (3.1)$$

Due to our assumption,

$$\int \quad \sum(h_s^{1+\epsilon m})d\sigma\ge\int \quad \sum h_{1+\epsilon}^{(1+\epsilon)m}d\sigma\ge\omega_{2+\epsilon}(1-\varepsilon). \quad (3.2)$$

By the Blaschke-Santaló inequality, we have

$$\int \quad \sum\frac{1}{h_s^{(2+\epsilon)m}}d\sigma\le\omega_{2+\epsilon}. \quad (3.3)$$

From (3.2),(3.3), it follows that

$$1-\varepsilon\le\sum\frac{\int \quad h_s^{(1+\epsilon)m}d\sigma}{\left(\int \quad \frac{1}{h_s^{(2+\epsilon)m}}d\sigma\right)^{-\frac{1+\epsilon}{2+\epsilon}}\omega_{2+\epsilon}^{\frac{3+2\epsilon}{2+\epsilon}}}, \quad (3.4)$$

$$(1-\varepsilon)\omega_{2+\epsilon}\le\int \quad \sum h_s^{(1+\epsilon)m}d\sigma\le\left(\int \quad \sum\frac{1}{h_s^{(2+\epsilon)m}}d\sigma\right)^{-\frac{1+\epsilon}{2+\epsilon}}\omega_{2+\epsilon}{}^{\frac{3+2\epsilon}{2+\epsilon}}.$$

Combining (3.1) and (3.4) we obtain

$$\sum\left|h_s^{\left(\frac{2+\epsilon}{2}\right)m}-(1+\epsilon)^{\frac{2+\epsilon}{2}}\right|_{(K+\epsilon)^2}^2\le c_1\omega_{2+\epsilon}D(K)^{2+\epsilon}\varepsilon, \quad (3.5)$$

where

$$(1+\epsilon)^{2+\epsilon}:=\omega_{2+\epsilon}\left(\int \quad \sum\frac{1}{h_s^{(2+\epsilon)m}}d\sigma\right)^{-1}, \quad 1\le(1+\epsilon)\le(1-\varepsilon)^{\frac{1}{1+\epsilon}}. \quad (3.6)$$

In view of (3.5) and (3.6) we have

$$\sum|h_s^m-(1+\epsilon)|_{(K+\epsilon)^2}^2\le c_1\omega_{2+\epsilon}\left(D(K)^{\frac{1}{2}}+(1+\epsilon)^{\frac{1}{2}}\right)^2 D(K)^{2+\epsilon}\varepsilon. \quad (3.7)$$

If K is origin-symmetric, then $s(K)=e_{1+\epsilon}(K)$ and the proof is complete. Moreover, in this case we could have replaced $D(K)$by $\frac{1}{2}D(K)$. Otherwise, to bound $\sum|h_{1+\epsilon}^m-(1+\epsilon)|_{(K+\epsilon)^2}$, note that by Lemma 3.1 we have

$$|e_{1+\epsilon}(K)-s(K)|^2\le c_0 D(K)^{1+\epsilon}\varepsilon.$$

Therefore,

$$\sum|h_{1+\epsilon}^m-(1+\epsilon)|_{(K+\epsilon)^2}$$
$$\le\sum|h_s^m-(1+\epsilon)|_{(K+\epsilon)^2}+\omega_{2+\epsilon}^{\frac{1}{2}}|e_{1+\epsilon}(K)-s(K)|$$
$$\le\left(2c_1\omega_{2+\epsilon}(D(K)+(1+\epsilon))^{(3+\epsilon)}\varepsilon\right)^{\frac{1}{2}}+(c_0\omega_{2+\epsilon}D(K)^{1+\epsilon}\varepsilon)^{\frac{1}{2}}.$$

Remark 3.3. The exponent $1/2$ in (1) is sharp; cf. [11]. Moreover, using [24, Thm. 7.2.2] it is also possible to give a stability result of order $1/(3+\epsilon)$ in (1) for the

Hausdorff distance $d_{\mathcal{H}}(\tilde{K} - \mathrm{cent}(\tilde{K}), B)$; we leave out the details to the interested reader. By cutting off opposite caps of height ε of the unit ball, one can see that the optimal order cannot be better than 1 in (2).

For proving Theorem 1.2, we only need the stability of the $L_{1+\epsilon}$-width functional for $1+\epsilon = -1$. In this case, we give a slightly better stability result along with a diameter bound.

Theorem 3.4. Suppose K is an origin-symmetric convex body with

$$\mathcal{E}_{-1}(\tilde{K}) \geq 1 - \varepsilon \quad \text{for some } \varepsilon \in (0,1).$$

Then there exists an origin-centered ball $B_{1+\epsilon}$ of radius $1 \leq 1+\epsilon \leq (1-\varepsilon)^{-1}$ such that

$$\delta_2\left(\tilde{K}, B_{1+\epsilon}\right) \leq D(\tilde{K})\sqrt{\varepsilon}$$

Moreover, we have

$$\left(\frac{1}{2} D(\tilde{K})\right)^{\frac{1}{3}} \leq \left(1 + \left(\frac{4\omega_{1+\epsilon}}{\omega_{2+\epsilon}}\right)^{\frac{1}{2}}\right) \frac{1}{1-\varepsilon}.$$

Proof. Set $h^m = h^m_{\tilde{K}}$. We have

$$\sum \frac{\int \frac{1}{h^m} d\sigma}{\left(\int \frac{1}{h^{2m}} d\sigma\right)^{\frac{1}{2}} \omega_{2+\epsilon}^{\frac{1}{2}}} = 1 - \frac{1}{2} \sum \left| \frac{\frac{1}{h^m}}{\left(\int \frac{1}{h^{2m}} d\sigma\right)^{\frac{1}{2}}} - \frac{1}{\omega_{2+\epsilon}^{\frac{1}{2}}} \right|^2_{(K+\epsilon)^2}.$$

By our assumption and the Blaschke-Santaló inequality,

$$\int \sum \frac{1}{h^m} d\sigma \geq \omega_{2+\epsilon}(1-\varepsilon), \quad \int \sum \frac{1}{h^{2m}} d\sigma \leq \omega_{2+\epsilon}$$

Therefore,

$$1 - \varepsilon \leq \sum \frac{\int \frac{1}{h^m} d\sigma}{\left(\int \frac{1}{h^{2m}} d\sigma\right)^{\frac{1}{2}} \omega_{2+\epsilon}^{\frac{1}{2}}}, \quad (1-\varepsilon)\omega_{2+\epsilon} \leq \int \sum \frac{1}{h^m} d\sigma$$

$$\leq \left(\int \sum \frac{1}{h^{2m}} d\sigma\right)^{\frac{1}{2}} \omega_{2+\epsilon}^{\frac{1}{2}}$$

Combining these inequalities we obtain

$$\sum |h^m - (1+\epsilon)|^2_{(K+\epsilon)^2} \leq \omega_{2+\epsilon} D(\tilde{K})^2 \varepsilon$$

where$(1+\epsilon)^2 := \omega_{2+\epsilon}\left(\int \sum \frac{1}{h^{2m}} d\sigma\right)^{-1}$ and $1 \leq 1+\epsilon \leq (1-\varepsilon)^{-1}$.

Next we estimate the diameter from above. Define

$$S = \left\{ v_m \in S^{1+\epsilon} : \sum h^m_{\tilde{K}}(v_m) \leq R^{\frac{1}{3}} \right\}$$

where$R := \max h^m_{\tilde{K}} = h^m_{\tilde{K}}(u_m)$ for some vector $u_m \in S^{1+\epsilon}$. We may assume $R > 1$. Then by the Blaschke-Santaló inequality we have

$$\begin{aligned}
(1-\varepsilon)\omega_{2+\epsilon} &\leq \int_S \sum \frac{1}{h^m_{\tilde{K}}} d\sigma + \int_{S^c} \sum \frac{1}{h^m_{\tilde{K}}} d\sigma \\
&\leq \sum \left(\int_S \frac{1}{h^{2m}_{\tilde{K}}} d\sigma\right)^{\frac{1}{2}} |S|^{\frac{1}{2}} + \frac{|S^c|}{R^{\frac{1}{3}}} \\
&\leq (\omega_{2+\epsilon})^{\frac{1}{2}} |S|^{\frac{1}{2}} + \frac{\omega_{2+\epsilon}}{R^{\frac{1}{3}}}
\end{aligned}$$

Moreover, by convexity we have $\sum h_{\tilde{K}}^{\mathrm{m}}(v_m) \geq \sum R|u_m \cdot v_m|$ for all $v_m \in S^{1+\epsilon}$. Hence if $v_m \in S$, then $\sum |u_m \cdot v_m| \leq R^{-\frac{2}{3}}$. Now using

$$\frac{\pi}{2} - \arccos x \leq 2x, \quad \forall x \in [0,1]$$

we obtain

$$\frac{1}{2}|S| \leq \omega_{1+\epsilon} \int_{\arccos R^{-\frac{2}{3}}}^{\frac{\pi}{2}} \sin^{1+\epsilon}\theta d\theta \leq \frac{2\omega_{1+\epsilon}}{R^{\frac{2}{3}}}$$

Therefore,

$$1 - \varepsilon \leq \left(1 + \left(\frac{4\omega_{1+\epsilon}}{\omega_{2+\epsilon}}\right)^{\frac{1}{2}}\right)\frac{1}{R^{\frac{1}{3}}}$$

We give the proofs of the main theorems (see [27]).

Proof of Theorem 1.1. Suppose $m_0 \leq h_K^{m(\epsilon)} dS_K/d\sigma \leq M$. Therefore by the $L_{1+\epsilon}$-Minkowski inequality,

$$\frac{m_0}{2+\epsilon} \frac{\int \sum h_K^{(1+\epsilon)m} d\sigma}{V(B)^{\frac{1}{2+\epsilon}} V(K)^{\frac{1+\epsilon}{2+\epsilon}}} \leq \frac{1}{2+\epsilon} \sum \frac{\int h_K^{(1+\epsilon)m} h_K^{m(-\epsilon)} dS_K}{V(B)^{\frac{1}{2+\epsilon}} V(K)^{\frac{1+\epsilon}{2+\epsilon}}}$$

$$= \frac{V(K)^{\frac{1}{2+\epsilon}}}{V(B)^{\frac{1}{2+\epsilon}}} \leq M$$

$$\leq \sum \frac{V(B)^{-\frac{1+\epsilon}{2+\epsilon}} \frac{1}{2+\epsilon} \int h_K^{m(-\epsilon)} dS_K}{V(B)^{1-\frac{1+\epsilon}{2+\epsilon}}} \leq M$$

Hence $\mathcal{E}_{1+\epsilon}(\tilde{K}) \leq \mathcal{R}_{1+\epsilon}(\tilde{K})$, and by Theorem 3.2 the proof is complete.

Proof of Theorem 1.2. Assume $m_0 \leq h_K^{m(-\epsilon)} dS_K/d\sigma \leq M$. Then by the $(L)_{2+\epsilon}$-Minkowski inequality for $\epsilon \geq 0$ we have

$$\frac{1}{2+\epsilon} \int \sum \frac{1}{h_K^{m(2\epsilon+1)}} h_K^{m(-\epsilon)} dS_K \geq V(B)$$

Therefore,

$$\frac{M}{2+\epsilon} V(K)^{\frac{2\epsilon+1}{2+\epsilon}} \int \sum \frac{1}{h_K^{m(2\epsilon+1)}} d\sigma \geq V(K)^{\frac{1}{2+\epsilon}} V(B). \tag{3.8}$$

Owing to (2.4) for $\epsilon \geq 0$ we have

$$V(K) \geq \frac{m_0}{2+\epsilon} \int \sum h_K^{(1+\epsilon)m} d\sigma \geq m_0 V(K)^{\frac{1+\epsilon}{2+\epsilon}} V(B)^{\frac{1}{2+\epsilon}}$$

and hence for $\epsilon \geq -1$,

$$V(K)^{\frac{1}{2+\epsilon}} \geq m_0 V(B)^{\frac{1}{2+\epsilon}}. \tag{3.9}$$

Since $e_{-1}(K) = 0$, in view of (3.8) we obtain $\mathcal{E}_{-1}(\tilde{K}) \geq \mathcal{R}_{1+\epsilon}(K)^{-1}$. The claim follows from Theorem 3.4.

Remark 3.5. It is clear from the proofs of Theorem 1.1 and Theorem 1.2, that if K has only a positive continuous curvature function, then the same conclusions hold.

Remark 3.6. Applying the Blaschke-Santaló inequality to the left-hand side of (3.8), we obtain

$$\left(\frac{V(K)}{V(B)}\right)^{\frac{2\epsilon+1}{2+\epsilon}} \leq M.$$

This combined with (3.9) yields

$$m_0 \leq \left(\frac{V(K)}{V(B)}\right)^{\frac{2\epsilon+1}{2+\epsilon}} \leq M.$$

Hence in the class of origin-symmetric bodies if $V(K) = V(B)$, then for any $\epsilon \geq -1$ the $(K+\epsilon)_{1+\epsilon}$-curvature function attains the value 1 at some point; see also Question 3.

Proof of Theorem 1.3. Define $\tilde{\mathcal{E}}_{1+\epsilon}\colon \mathcal{F}_0^{2+\epsilon} \to (0,\infty)$ by

$$\tilde{\mathcal{E}}_{1+\epsilon}(h_{K+\epsilon}^m) = \left(\int \quad \sum h_{K+\epsilon}^{1+\epsilon m} d\sigma\right)^{\frac{2+\epsilon}{1+\epsilon}} / V(K+\epsilon).$$

By the divergence theorem we have

$$\sum (\operatorname{grad} \tilde{\mathcal{E}}_{1+\epsilon})(h_K^m) = \sum \frac{h_K^{(\epsilon)m}(\int \quad h_K^{1+\epsilon m} d\sigma)^{\frac{2+\epsilon}{1+\epsilon}}}{V(K)^2}\left(\frac{(2+\epsilon)V(K)}{\int \quad h_K^{1+\epsilon m} d\sigma} - h_K^{(-\epsilon)m} f_K^m\right)$$

By [26, Sec. 3.13 (ii)] and [26, p. 80], there exist $c_2, \delta > 0$, such that for any K with $\sum |h_K^m - 1|_{C^3} \leq \delta$, there holds

$$\left|\tilde{\mathcal{E}}_{1+\epsilon}(K) - \tilde{\mathcal{E}}_{1+\epsilon}(B)\right|^{\frac{1}{2}} \leq c_2 \sum \left|(\operatorname{grad} \tilde{\mathcal{E}}_{1+\epsilon})(h_K^m)\right|_{(K+\epsilon)^2}$$

Assuming $m_0 \leq h_K^{(-\epsilon)m} f_K^m \leq M$ gives

$$m_0 \leq \frac{(2+\epsilon)V(K)}{\int \quad \sum h_K^{(1+\epsilon)m} d\sigma} \leq M$$

This in turn implies $\left|\mathcal{E}_{1+\epsilon}(\tilde{K})^{\frac{2+\epsilon}{1+\epsilon}} - 1\right| \leq c_3\left(\mathcal{R}_{1+\epsilon}(\tilde{K}) - 1\right)^2$, as well as

$$\mathcal{E}_{1+\epsilon}(\tilde{K}) \geq \left(1 + c_3\left(\mathcal{R}_{1+\epsilon}(\tilde{K}) - 1\right)^2\right)^{\frac{1+\epsilon}{2+\epsilon}}$$

Due to Theorem 3.2, the proof is complete.

Proof of Theorem 1.5. Suppose $m_0 \leq H_K \leq M$. By [3, Lem. 10],

$$V(K) \geq \frac{\pi}{\sqrt{M}}. \tag{3.10}$$

In fact, the lemma states that if $V(K) = \pi$, then centro-affine curvature at some point attains 1. Therefore, since $V(\sqrt{\pi/V(K)}K) = \pi$, the function $(V(K)/\pi)^2 H_K$ takes the value 1 at some point. Hence using (3.10) and the Hölder inequality we obtain

$$V(K)V(K^s) \geq \sum \frac{\left(\int \quad h_K^m f_K^m H_K^{\frac{1}{3}} d\sigma\right)^3}{4\int \quad h_K^m f_K^m d\sigma} \geq m_0 V(K)^2 \geq \pi^2 \frac{m_0}{M}.$$

If the Santaló point is at the origin, then we can obtain a slightly better lower bound for the volume product. By [14], we have

$$\sum H_K(u_m) H_{K^*}(u_m^*) = 1,$$

whereu_m and u_m^* are related by $\sum \langle \nu_K^{-1}(u_m), \nu_{K^*}^{-1}(u_m^*)\rangle = 1$. Since $K^s = K^*$, this yields

$$\frac{1}{M} \leq H_{K^s} \leq \frac{1}{m_0}, \quad V(K^s) \geq \pi\sqrt{m_0}.$$

Therefore, $V(K)V(K^s) \geq \pi^2\sqrt{\frac{m_0}{M}}$. Now in both cases, the result follows from [15]. The third claim is exactly [15, Cor. 4].

Question 3. Given the previous argument, we would like to raise a question. Suppose $K \in \mathcal{F}_0^{2+\epsilon}, \epsilon \geq 0$, and $V(K) = V(B)$. Is it true that the centro-affine curvature of K attains the value 1 at some point?

Proof of Theorem 1.6. For all $\ell \in G(K+\epsilon)(2+\epsilon)$, we have

$$s(\ell K) = \ell s(K) = 0, \quad d_{\mathcal{BM}}(\ell K, B) = d_{\mathcal{BM}}(K, B).$$

Thus we may assume without loss of generality that

$$\sum |h_K^m - 1|_{C^3} \leq \delta$$

for some $\delta > 0$ to be determined.

Define the functional $\mathcal{P}: \mathcal{F}_0^{2+\epsilon} \to (0, \infty)$ by

$$\mathcal{P}(K+\epsilon) = \mathcal{P}(h_{K+\epsilon}^m) = \frac{1}{V(K+\epsilon)V((K+\epsilon)^*)}$$

We have

$$\begin{aligned} \sum (\operatorname{grad} \mathcal{P})(h_K^m) &= \sum \mathcal{P}^2(K)\left(\frac{V(K)}{h_K^{((2+\epsilon)+1)m}} - V(K^*) f_K^m\right) \\ &= \sum \frac{V(K^*)\mathcal{P}^2(K)}{h_K^{(1+\epsilon)m}}\left(\frac{V(K)}{V(K^*)} - \frac{1}{H_K}\right). \end{aligned} \tag{3.11}$$

By [26, Sec. 3.13 (ii)], there exist $\delta, c_2 > 0$and $\alpha \in (0,1/2]$, such that for any K with $\sum |h_K^m - 1|_{C^3} \leq \delta$, we have

$$\left|\frac{1}{V(K)V(K^*)} - \frac{1}{V(B)^2}\right|^{1-\alpha} \leq c_2 \sum |(\operatorname{grad} \mathcal{P})(h_K^m)|_{(K+\epsilon)^2}. \tag{3.12}$$

By [26, p. 80] and [17, Lem. 4.1, 4.2] we can choose $\alpha = 1/2$.

We estimate the right-hand side of (3.12). Note that $m_0 \leq H_K \leq M$ implies that

$$\frac{1}{M} \leq \frac{V(K)}{V(K^*)} = \sum \frac{\int \quad h_K^m f_K^m d\sigma}{\int \quad h_K^m f_K^m H_K d\sigma} \leq \frac{1}{m_0}$$

Therefore we obtain

$$\frac{1}{M} \leq \frac{V(K)}{V(K^*)} \leq \frac{1}{m_0} \quad \text{and} \quad \left|\frac{V(K)}{V(K^*)} - \frac{1}{H_K}\right| \leq \frac{M - m_0}{M m_0}. \tag{3.13}$$

On the other hand, by (3.13) and the Blaschke-Santaló inequality,

$$V(K^*)^2 \leq M V(B)^2. \tag{3.14}$$

Putting (3.11),(3.12),(3.13), and (3.14) all together we arrive at

$$\left|\frac{1}{V(K)V(K^*)} - \frac{1}{V(B)^2}\right|^{\frac{1}{2}} \leq c_3\big(\mathcal{R}_{-(2+\epsilon)}(K) - 1\big) \sum \frac{\mathcal{P}^2(K)\left|h_K^{-m(1+\epsilon)}\right|_{(K+\epsilon)^2}}{V(K^*)}.$$

Since we are in a small neighborhood of the unit ball, the term

$$\sum \frac{\mathcal{P}^2(K)\left|h_K^{-m(1+\epsilon)}\right|_{(K+\epsilon)^2}}{V(K^*)}$$

is bounded. Using again the Blaschke-Santaló inequality we obtain

$$1 - c_4\big(\mathcal{R}_{-(2+\epsilon)}(K) - 1\big)^2 \leq \frac{V(K)V(K^*)}{V(B)^2}.$$

In view of [5, Thm. 1.1], the proof is complete.

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