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The Operators of Composition-Differentiation on the Hardy Space

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Abstract.
Suppose φ_j be a nonconstant analytic self-map of the open unit disk in $\mathbb{C}$, with $\sum\|\varphi_j\|_\infty < 1$. Regard the operators D_{φ_j}, acting on the Hardy space H^2,given by differentiation followed by composition with φ_j. Mahsa Fatehi and Christopher N. B. Hammond. Obtain results relating to the adjoint, norm, and spectrum of such an operator.

Key words: Operator, Differentiation, Hardy Space, analytic functions, composition–differentiation, open unit disk

1. Preliminaries

Suppose $\mathbb{D}$ denote the open unit disk in the complex plane. The Hardy space H^2 is the Hilbert space consisting of every analytic function $\sum f_j(z) = \sum_{n=0}^{\infty} \sum a_n^j z^n$ on $\mathbb{D}$ such that

$$\sum\|f_j\| = \sum_{n=0}^{\infty} \sum |a_n^j|^2 < \infty.$$

For $\sum f_j(z) = \sum_{n=0}^{\infty} \sum a_n^j z^n$ and $\sum g^j(z) = \sum_{n=0}^{\infty} \sum b_n^j z^n$ in H^2, their inner product is defined

$$\sum\langle f_j, g^j\rangle = \sum_{n=0}^{\infty} \sum a_n^j \overline{b_n^j}.$$

Write H^∞ to denote the space of every bounded analytic function on $\mathbb{D}$, with $\sum\|f_j\|_\infty = sup\{\sum|f_j(z)| : z \in \mathbb{D}\}$.

For an analytic map $\varphi_j : \mathbb{D} \to \mathbb{D}$, the composition operators C_{φ_j} are defined

$$C_{\varphi_j}(f_j) = f_j \circ \varphi_j.$$

Every composition operators are bounded on H^2, with $\sum \sqrt{\frac{1}{1-|\varphi_j(0)|^2}} \le \sum \|C_{\varphi_j}\| \le \sum \sqrt{\frac{1+|\varphi_j(0)|}{1-|\varphi_j(0)|}}$. (See, for example, [2, Corollary 3.7].) For a function ψ_j in H^∞, the Toeplitz operators $T_{\psi_j}^j$ are defined $T_{\psi_j}^j(f_j) = \psi_j \cdot f_j$.

Every such operators are bounded on H^2, with $\|T_{\psi_j}^j\| = \|\psi_j\|_\infty$(see [6, Theorem5]). Abusing notation somewhat, we will write T_z^j to indicate the operators $T_{\psi_j}^j$ for $\psi_j(z) = z$. Observe that T_z^j is an isometry on H^2.

In the context of analytic functions on $\mathbb{D}$, it also appears reasonable to peeking operators defined in terms of differentiation. It is easy to see that the differentiation operators $D(f_j) = \acute{f}_j$ are unbounded on the Hardy space: $\|D(z^n)\|/\|z^n\| = n$ for any natural number n. Nevertheless, for many analytic maps $\varphi_j : \mathbb{D} \to \mathbb{D}$, the operators

$$f_j(z) \mapsto \acute{f}_j\left(\varphi_j(z)\right) \quad (1.1)$$

is bounded on H^2. Some authors, following the example of [4] and [5], have used the notation $C_{\varphi_j} D$ to denote such operators. Because of the unboundedness of D,it makes sense to write (1.1) as a single operator, specifically when that operator is bounded on H^2. Write D_{φ_j} to denote the operators on H^2 given by $D_{\varphi_j}(f_j) = \acute{f}_j \circ \varphi_j$

In this paper we will refer to such an operator as a composition–differentiation operator. The Closed Graph Theorem exhibits that D_{φ_j} are bounded on H^2 whenever D_{φ_j} takes H^2 into itself.

Ohno [5] established a basic set of results relating to when the operators we are calling D_{φ_j} are bounded or compact on H^2. We will only be considering φ_j with $\sum\|\varphi_j\|_{\infty} < 1$, in which case D_{φ_j} is guaranteed to be Hilbert–Schmidt on H^2, and therefore both bounded and compact (see [5, Theorem 3.3]). There are cases of bounded or compact D_{φ_j} with $\sum\|\varphi_j\|_{\infty} = 1$, but they are beyond the scope of the current investigation.

The purpose of this note is to explore the operators D_{φ_j} in more detail. In specific, we find a representation for the adjoint $D^*_{\varphi_j}$ when φ_j is linear fractional (Theorem 1). In the particular case where $\rho_j(z) = r_j z$ for $0 < |r_j| < 1$, we compute the norm $\|D_{\rho_j}\|$ explicitly (Theorem 2) and exhibit that D_{ρ_j} is quasinilpotent (Proposition3). Applying established results relating to composition operators, also obtain estimates for the norm of D_{φ_j} whenever $\sum\|\varphi_j\|_{\infty} < 1$ (Proposition 4).

For any point w_j in $\mathbb{D}$, define $K_{w_j}(z) = \frac{1}{1-\overline{w_j}z}$.

It is well known that K_{w_j} acts as the reproducing kernel function for point-evaluation:

$$\sum \langle f_j, K_{w_j}\rangle = \sum f_j(w_j)$$

for any f_j in H^2. Therefore $\sum \left\|K_{w_j}\right\| = \sum \sqrt{K_{w_j}(w_j)} = \sum \sqrt{\frac{1}{1-|w_j|^2}}$.

In Identical manner, define $K^{(1)}_{w_j}(z) = \frac{z}{\left(1-\overline{w_j}z\right)^2}$.

Notice that $K^{(1)}_{w_j}$ acts as the reproducing kernel for point-evaluation of the first derivative:

$$\sum \langle f_j, K^{(1)}_{w_j}\rangle = \sum f_j'(w_j)$$

for any f_j in H^2 (see [2, Theorem 2.16]). In specific,

$$\sum \left\|K^{(1)}_{w_j}\right\| = \sum \sqrt{\left(K^{(1)}_{w_j}\right)'(w_j)} = \sum \sqrt{\frac{1+|w_j|^2}{\left(1-|w_j|^2\right)^3}}$$

for any w_j in $\mathbb{D}$.

It is well known that $C^*_{\varphi_j}\left(K_{w_j}\right) = K_{\varphi_j(w_j)}$ and $\left(T^j_{\psi_j}\right)^*\left(K_{w_j}\right) = \overline{\psi_j(w_j)}K_{w_j}$ for any w_j in $\mathbb{D}$.Identical property holds for any bounded operators D_{φ_j} on H^2. Observe that

$$\sum \langle f_j, D^*_{\varphi_j}\left(K_{w_j}\right)\rangle = \sum \langle D_{\varphi_j}(f_j), K_{w_j}\rangle = \sum f'\left(\varphi_j(w_j)\right) = \sum \langle f_j, K^{(1)}_{\varphi_j(w_j)}\rangle$$

for every f_j in H^2, so $D^*_{\varphi_j}\left(K_{w_j}\right) = K^{(1)}_{\varphi_j(w_j)}$ for any w_j in $\mathbb{D}$. In specific,

$$\sum \frac{\left\|D^*_{\varphi_j}(K_{w_j})\right\|}{\left\|K_{w_j}\right\|} = \sqrt{\sum \frac{\left(1-|w_j|^2\right)\left(1+|\varphi_j(w_j)|^2\right)}{\left(1-|\varphi_j(w_j)|^2\right)^3}}. \qquad (1.2)$$

Therefore, if D_{φ_j} is bounded on H^2, we see that

$$\sum \left\|D_{\varphi_j}\right\| \geq \sup_{w_j\in\mathbb{D}} \sqrt{\sum \frac{\left(1-|w_j|^2\right)\left(1+|\varphi_j(w_j)|^2\right)}{\left(1-|\varphi_j(w_j)|^2\right)^3}} \geq \sup_{w_j\in\mathbb{D}} \sqrt{\sum \frac{1-|w_j|^2}{\left(1-|\varphi_j(w_j)|^2\right)^3}}$$

And$\sum \left\| D_{\varphi_j} \right\|_e \geq \limsup_{|w_j|\to 1} \sqrt{\sum \frac{\left(1-|w_j|^2\right)\left(1+|\varphi_j(w_j)|^2\right)}{\left(1-|\varphi_j(w_j)|^2\right)^3}} \geq \limsup_{|w_j|\to 1} \sqrt{\sum \frac{1-|w_j|^2}{\left(1-|\varphi_j(w_j)|^2\right)^3}}$.(1.3)

These findings should be compared to [5, Corollary 3.2], which outlines the necessary and sufficient requirements for D_{φ_j} to be bounded or compact when φ_j is univalent. Inspecific, (1.3) exhibits that D_{φ_j}cannot be bounded if $\varphi_j$has finite angular derivativeat any point on $\partial\mathbb{D}$ (see [2, Theorem 2.44]). Taking $w_j = 0$ in (1.2), we see that $\sum \left\| D_{\varphi_j} \right\| \geq \sqrt{\sum \frac{1+|\varphi_j(0)|^2}{\left(1-|\varphi_j(0)|^2\right)^3}}$,

which exhibits that $\sum \left\| D_{\varphi_j} \right\| \geq 1$ for each $\varphi_j$and that $\sum \left\| D_{\varphi_j} \right\| > 1$ whenever $\varphi_j(0) \neq 0$.

For the duration of this note, we will assume that $\varphi_j$is a nonconstant analytic map with$\sum \|\varphi_j\|_\infty < 1$. As mentioned earlier, this assumption guarantees that $D_{\varphi_j}$is compact on H^2.

2. **Adjoints, Norms, and Spectra**

We obtain information about the adjoint, norm, and spectrum of D_{φ_j}in certain specific cases. If $\varphi_j(z) = \frac{a^j z + b^j}{c^j z + d^j}$

is a non constant linear fractional self-map of $\mathbb{D}$, hence the map $\sigma_j(z) = \frac{\overline{a^j} z - \overline{c^j}}{-\overline{b^j} z + \overline{d^j}}$ also takes $\mathbb{D}$ into itself (see [1, Lemma 1]). It is not difficult to exhibit that $\sum \|\sigma_j\|_\infty < 1$whenever $\sum \|\varphi_j\|_\infty < 1$. The link between these two maps has long been regarded in reference to the adjoints of composition operators.

Theorem 1. For a pair of linear fractional maps $\varphi_j$and σ_j, as described above,

$$D^*_{\varphi_j}\left(T^j_{K^{(1)}_{\sigma_j(0)}}\right)^* = T^j_{K^{(1)}_{\varphi_j(0)}} D_{\sigma_j}$$

Proof. Observe that $K^{(1)}_{\varphi_j(0)}(z) = \frac{z}{\left(1-(\overline{b^j}/\overline{d^j})z\right)^2} = \frac{(\overline{d^j})^2 z}{\left(-\overline{b^j}z+\overline{d^j}\right)^2}$

And $K^{(1)}_{\sigma_j(0)}(z) = \frac{z}{\left(1+(c^j/d^j)z\right)^2} = \frac{\left(d^j\right)^2 z}{\left(c^j z+d^j\right)^2}$.

Furthermore, see that; $T^j_{K^{(1)}_{\varphi_j(0)}} D_{\sigma_j}\left(K_{w_j}\right)(z) = T^j_{K^{(1)}_{\varphi_j(0)}} \frac{w_j}{\left(1-\overline{w_j}\left(\frac{\overline{a^j}z-\overline{c^j}}{-\overline{b^j}z+\overline{d^j}}\right)\right)^2}$

$$= \frac{(\overline{d^j})^2 z}{\left(-\overline{b^j}z + \overline{d^j}\right)^2} \cdot \frac{\overline{w_j}}{\left(1 - \overline{w_j}\left(\frac{\overline{a^j}z - \overline{c^j}}{-\overline{b^j}z + \overline{d^j}}\right)\right)^2}$$

$$= \frac{\overline{(d^j)^2 w_j} z}{\left(-\overline{b^j}z + \overline{d^j} - \overline{a^j}\,\overline{w_j}z + \overline{c^j}\,\overline{w_j}\right)^2}$$

$$= \frac{\overline{(d^j)^2 w_j}}{\left(\overline{c^j w_j} + \overline{d^j}\right)^2} \cdot \frac{z}{\left(1 - \left(\frac{\overline{a^j w_j} + \overline{b^j}}{\overline{c^j w_j} + \overline{d^j}}\right)z\right)^2}$$

for any $w_j$in $\mathbb{D}$. Second hand,

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$$D_{\varphi_j}^{*}\left(T^{j}_{K^{(1)}_{\sigma_j(0)}}\right)^{*}\left(K_{w_j}\right)=\frac{\overline{(d^J)^2 w_j}}{\overline{(c^J w_j+d^J)}^2}D_{\varphi_j}^{*}\left(K_{w_j}\right)=\frac{\overline{(d^J)^2 w_j}}{\overline{(c^J w_j+d^J)}^2}K^{(1)}_{\varphi_j(w_j)}.$$

Therefore, $D_{\varphi_j}^{*}\left(T^{j}_{K^{(1)}_{\sigma_j(0)}}\right)^{*}$ and $T^{j}_{K^{(1)}_{\varphi_j(0)}}D_{\sigma_j}$ agree on the span of the reproducing kernelfunctions, which constitutes a dense subset of H^2. Therefore, the two operators are identical on H^2.

This result bears a close resemblance to Cowen's adjoint formula for composition operators (see [1, Theorem 2]), which can be rewritten $C_{\varphi_j}^{*}\left(T^{j}_{K^{(1)}_{\sigma_j(0)}}\right)^{*}=T^{j}_{K_{\varphi_j(0)}}C_{\sigma_j}$.

Let us focus on the specific situation where $\rho_j(z)=r_j z$ for a real number r_j. In this case, Theorem 1 reduces to $D_{\rho_j}^{*}\left(T_z^{j}\right)^{*}=T_z^{j}D_{\rho_j}$. Therefore

$$D_{\rho_j}^{*}=D_{\rho_j}^{*}\left(T_z^{j}\right)^{*}T_z^{j}=T_z^{j}D_{\rho_j}T_z^{j},$$

and therefore $D_{\rho_j}D_{\rho_j}^{*}=D_{\rho_j}T_z^{j}D_{\rho_j}T_z^{j}=\left(D_{\rho_j}T_z^{j}\right)^2$.

Note that; $\left(D_{\rho_j}T_z^{j}\right)^{*}=\left(T_z^{j}\right)^{*}D_{\rho_j}^{*}=\left(T_z^{j}\right)^{*}T_z^{j}D_{\rho_j}T_z^{j}=D_{\rho_j}T_z^{j}$,

so $D_{\rho_j}T_z^{j}$ is self-adjoint. If $0<r_j<1$, we will see that $D_{\rho_j}T_z^{j}=\left|D_{\rho_j}^{*}\right|$. In specific,if we can determine the spectrum of $D_{\rho_j}T_z^{j}$, we will be able to compute $\left\|D_{\rho_j}\right\|$.

Theorem 2. If $\rho_j(z)=r_j z$ for some real number $0<r_j<1$, hence

$$\sum\left\|D_{\rho_j}\right\|=\sum\left\lfloor\frac{1}{1-r_j}\right\rfloor r_j^{\lfloor 1/(1-r_j)\rfloor-1}, \qquad 2.1$$

where $\lfloor\cdot\rfloor$ denotes the greatest integer function.

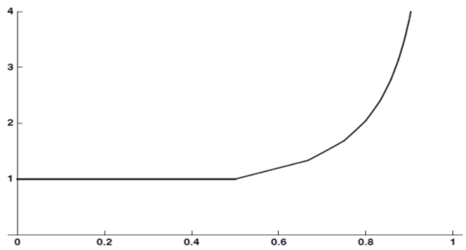

Figure 1. The quantity $\sum\left\|D_{\rho_j}\right\|$ for different values of r_j.

Proof. Note that $D_{\rho_j}T_z^{j}(z^{n-1})=D_{\rho_j}(z^n)=n(r_j z)^{n-1}=nr_j^{n-1}z^{n-1}$

for any natural number n. Therefore, the spectrum of $D_{\rho_j}T_z^{j}$ includes $\{nr_j^{n-1}: n\in\mathbb{N}\}$.

Second hand, suppose λ_j be an arbitrary eigenvalue for $D_{\rho_j}T_z^{j}$ with corresponding eigenvector f_j. Observe that; $\lambda_j f_j(z)=f_j(r_j z)+r_j z f_j{}'(r_j z)$.

If $f_j(0)\neq 0$, hence $\lambda_j=1$. If $f_j(0)=0$, differentiate both sides of the equation above to obtain $\lambda_j f_j{}'(z)=2r_j f_j{}'(r_j z)+r_j^2 z f_j{}''(r_j z)$.

If $f_j{}'(0)\neq 0$, hence $\lambda_j=2r_j$. Differentiating a total of $n-1$ times, we obtain

$$\lambda_j f_j^{(n-1)}(z)=nr_j^{n-1}f_j^{(n-1)}(r_j z)+r_j^n z f_j^{(n)}(r_j z).$$

If $f_j^{(n-1)}(0)\neq 0$, Therefore $\lambda_j=nr_j^{n-1}$. Therefore, any eigenvalue must have the form nr_j^{n-1} for some natural number n. Since $D_{\rho_j}T_z^{j}$ is compact, its spectrum is exactly $\{0\}\cup\{nr_j^{n-1}: n\in\mathbb{N}\}$.

Since all the eigenvalues are positive, see that $\sum D_{\rho_j} T_z^j = \sum \left| D_{\rho_j}^* \right|$. Therefore,
$\sum \left\| D_{\rho_j} \right\| = max \left\{ \sum n r_j^{n-1} : n \in \mathbb{N} \right\}$.
To obtain (2.1), observe that the function $h_j(x) = x r_j^{n-1}$ has precisely one local extremum on $[0, \infty)$, which are also its absolute maximum. Therefore we simply need to find the greatest natural number n so that $(n-1) r_j^{n-2} \le n r_j^{n-1}$
or equivalently $1 - \frac{1}{n} \le r_j$
or $n \le \frac{1}{1-r_j}$.
Thus the quantity $n r_j^{n-1}$ is maximized when $n = \lfloor 1/(1-r_j) \rfloor$.
There are several interesting consequences of Theorem 2. First of each, $\sum \left\| D_{\rho_j} \right\| = 1$ for $0 < r_j \le 1/2$ and $\sum \left\| D_{\rho_j} \right\| > 1$ for $1/2 < r_j < 1$. Secondly, $\sum \left\| D_{\rho_j} \right\|$ tends to ∞ as r_j goes to 1. Since composition with a rotation is an isometry, (2.1) holds with r_j replaced by $|r_j|$ for any complex number r_j with $0 < \sum |r_j| < 1$. Likewise, the same formula holds for $\sum \left\| D_{\varphi_j} \right\|$ where $\varphi_j(z) = r_j z^k$ for any k in $\mathbb{N}$.
We deduce this section with an observation relating to the spectrum of D_{ρ_j}, which indeed pertains to a slightly larger class of operators.

Proposition 3. If $\varphi_j(z) = a^j z + b^j$, where $0 < \sum |a^j| < 1 - \sum |b^j|$, the operators D_{φ_j} are quasi nilpotent; that is, its spectrum is $\{0\}$.

Proof. Assume that λ_j are a nonzero element of the spectrum of D_{φ_j}, which would necessarily be an eigenvalue. Hence there is a nonzero function f_j in H^2 such that;

$$\lambda_j f_j(z) = f_j'(a^j z + b^j).$$

It is clear from this formula that f_j cannot be a polynomial. Differentiating n times,see that

$$\lambda_j f_j^{(n)}(z) = (a^j)^n f_j^{(n+1)}(a^j z + b^j),$$

so that $(\lambda_j/(a^j)^n) f_j^{(n)}(z) = \left(f_j^{(n)}\right)'(a^j z + b^j)$.
Since $f_j^{(n)}$ are not identically 0, we deduce that $\lambda_j/(a^j)^n$ must be an eigenvalue for D_{φ_j} for ever yn, which is impossible. Therefore $\lambda_j = 0$ are the only elements in the spectrum of D_{φ_j}.

Remark. Make a stronger statement than Proposition 3. When $\varphi_j(z) = a^j z + b^j$, note that $D_{\varphi_j}^n$ are never equal to the trivial operators on H^2. Furthermore, for any natural number M, the closed linear span of the set $\{z^n\}_{n=0}^M$ is invariant under the operators D_{φ_j}. Therefore [3, Corollary 1] exhibits that D_{φ_j} are a compact universal quasi nilpotent operators; that is, the norm closure of $\left\{ W D_{\varphi_j} W^{-1}\right.$: W is an invertible operator on $\left. H^2 \right\}$ contains every compact quasi nilpotent operator.

3. Connections with composition operators

We can view the operators D_{φ_j} as the product $C_{\varphi_j} D$, although the benefit of this representation is limited by the fact that D is unbounded on H^2. The bounded operators D_{ρ_j}, where $\rho_j(z) = r_j z$ for $0 < r_j < 1$, can act as asurrogate for D in the situation where $\sum \left\| \varphi_j \right\|_\infty < 1$. Recall that D_{ρ_j} are compact and quasi nilpotent, and that we know both its adjoint and its norm.

Take $\sum\|\varphi_j\|_\infty \le \sum r_j < 1$ and define $(\varphi_j)_{r_j} = (1/r_j)\varphi_j$. Observe that $D_{\varphi_j} = C_{(\varphi_j)_{r_j}} D_{\rho_j}$.
(3.1)

Since $\sum \left\|D_{\varphi_j}\right\| = \sum \left\|C_{(\varphi_j)_{r_j}}\right\| \left\|D_{\rho_j}\right\|$, obtain the following estimate for $\left\|D_{\varphi_j}\right\|$.

Proposition 4. If φ_j are a nonconstant analytic self-map of $\mathbb{D}$ with $\sum\|\varphi_j\|_\infty < 1$, then

$$\sum \sqrt{\frac{1+|\varphi_j(0)|^2}{\left(1-|\varphi_j(0)|^2\right)^3}} \le \sum \left\|D_{\varphi_j}\right\| \le \sum \sqrt{\frac{r_j+|\varphi_j(0)|}{r_j-|\varphi_j(0)|}} \left[\frac{1}{1-r_j}\right] r_j^{\lfloor 1/1-r_j \rfloor -1}$$

whenever $\sum\|\varphi_j\|_\infty \le \sum r_j < 1$.

If $\sum\|\varphi_j\|_\infty \le 1/2$ we may take $r_j = 1/2$ to see that $\sum \sqrt{\frac{1+|\varphi_j(0)|^2}{\left(1-|\varphi_j(0)|^2\right)^3}} \le \sum \left\|D_{\varphi_j}\right\| \le \sum \sqrt{\frac{1+2|\varphi_j(0)|}{1-2|\varphi_j(0)|}}$.

In specific, $\sum \left\|D_{\varphi_j}\right\| = 1$ whenever both $\sum\|\varphi_j\|_\infty \le 1/2$ and $\varphi_j(0) = 0$.

Example 5. Suppose $\varphi_j(z) = a^j z + b^j$, where $a^j \ne 0$ and $\sum|a^j| + \sum|b^j| \le 1/2$, so that

$$\sum \sqrt{\frac{1+|b^j|^2}{\left(1-|b^j|^2\right)^3}} \le \sum \left\|D_{\varphi_j}\right\| \le \sum \sqrt{\frac{1+2|b^j|}{1-2|b^j|}}.$$

We can indeed improve on Proposition 4 because we know the norm of $C_{(\varphi_j)_{r_j}}$ precisely. For $r_j = 1/2$, it follows from [1, Theorem 3] that

$$\sum \left\|C_{(\varphi_j)_{r_j}}\right\| = \sum \sqrt{\frac{2}{1+4|a^j|^2-4|b^j|^2+\sqrt{\left(1-4|a^j|^2+4|b^j|^2\right)^2-16|b^j|^2}}}.$$

This value serves as a more precise upper bound for $\left\|D_{\varphi_j}\right\|$.

Working with D_{ρ_j} can also be useful for describing the adjoint of operators D_{φ_j}.

Collecting (3.1) with the identity $D_{\rho_j}^* = T_{\bar z}^j D_{\rho_j} T_{\bar z}^j$, we see that $D_{\varphi_j}^* = T_{\bar z}^j D_{\rho_j} T_{\bar z}^j C_{(\varphi_j)_{r_j}}^*$ whenever $\sum\|\varphi_j\|_\infty \le \sum r_j < 1$. This annotation allows us to determine $D_{\varphi_j}^*$ whenever we know $C_{(\varphi_j)_{r_j}}^*$.

Example 6. Suppose $\varphi_j(z) = r_j z^2$ for $0 < r_j < 1$ and peeking $(\varphi_j)_{r_j}(z) = z^2$. It is well known (see [2, Exercise 9.1.1]) that $C_{(\varphi_j)_{r_j}}^*(f_j) = \frac{f_j(\sqrt{z})+f_j(-\sqrt{z})}{2}$ for any f_j in H^2; that is, $C_{(\varphi_j)_{r_j}}^*(z^{2n}) = z^n$ and $C_{(\varphi_j)_{r_j}}^*(z^{2n+1}) = 0$ for any nonnegative integer n. Consequently, $D_{\varphi_j}^*(z^{2n}) = (n+1) r_j^n z^{n+1}$ and $D_{\varphi_j}^*(z^{2n+1}) = 0$. For example, note that

$$\sum D_{\varphi_j}^*(\log(1-z)) = \sum D_{\varphi_j}^*\left(-\sum_{n=1}^{\infty}\frac{z^n}{n}\right) = -\sum\sum_{n=1}^{\infty}\frac{(n+1)r_j^n z^{n+1}}{2n}$$

$$= -\frac{z}{2}\sum\sum_{n=1}^{\infty}(r_j z)^n - \frac{z}{2}\sum\sum_{n=1}^{\infty}\frac{(r_j z)^n}{n}$$

$$= -\sum\frac{r_j z^2}{2(1-r_j z)} + \sum\frac{z\log(1-r_j z)}{2}.$$

Observing that every D_{φ_j} from Example 6 has at least one nonzero eigenvalue, since

$$D_{\varphi_j}(z^2) = 2(r_j z^2) = (2r_j)z^2.$$

Results:

1. Ohno [5] established a basic set of results relating to when the operators we are calling D_{φ_j} are bounded or compact on H^2.
2. Applying established results relating to composition operators, also obtain estimates for the norm of D_{φ_j} whenever $\sum\|\varphi_j\|_\infty < 1$ (Proposition 4).
3. This result bears a close resemblance to Cowen's adjoint formula for composition operators (see [1, Theorem 2]), which can be rewritten $C^*_{\varphi_j}\left(T^j_{K^{(1)}_{\sigma_j(0)}}\right)^* = T^j_{K_{\varphi_j(0)}} C_{\sigma_j}$.

Conclusion

In order to determine if the operators we are calling D_{φ_j} are bounded or compact on H^2, conclusion is quite similar to Cowen's adjoint formula for composition operators, which can be expressed as $C^*_{\varphi_j}\left(T^j_{K^{(1)}_{\sigma_j(0)}}\right)^* = T^j_{K_{\varphi_j(0)}} C_{\sigma_j}$.

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